What mathematics is hidden behind the astronomical clock of Prague?

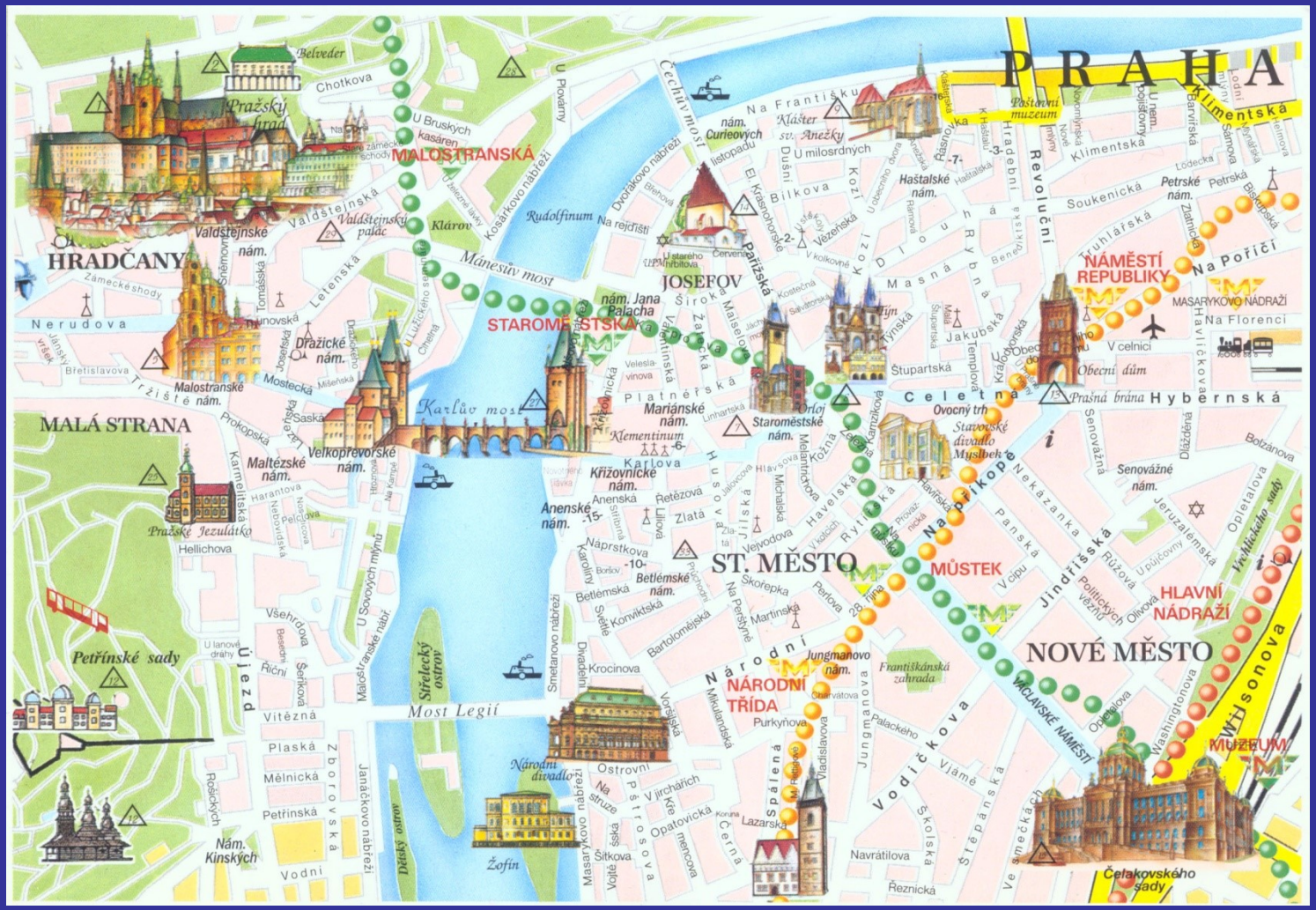
Construction of Šindel sequences

Michal Křížek
Alena Šolcová
Lawrence Somer

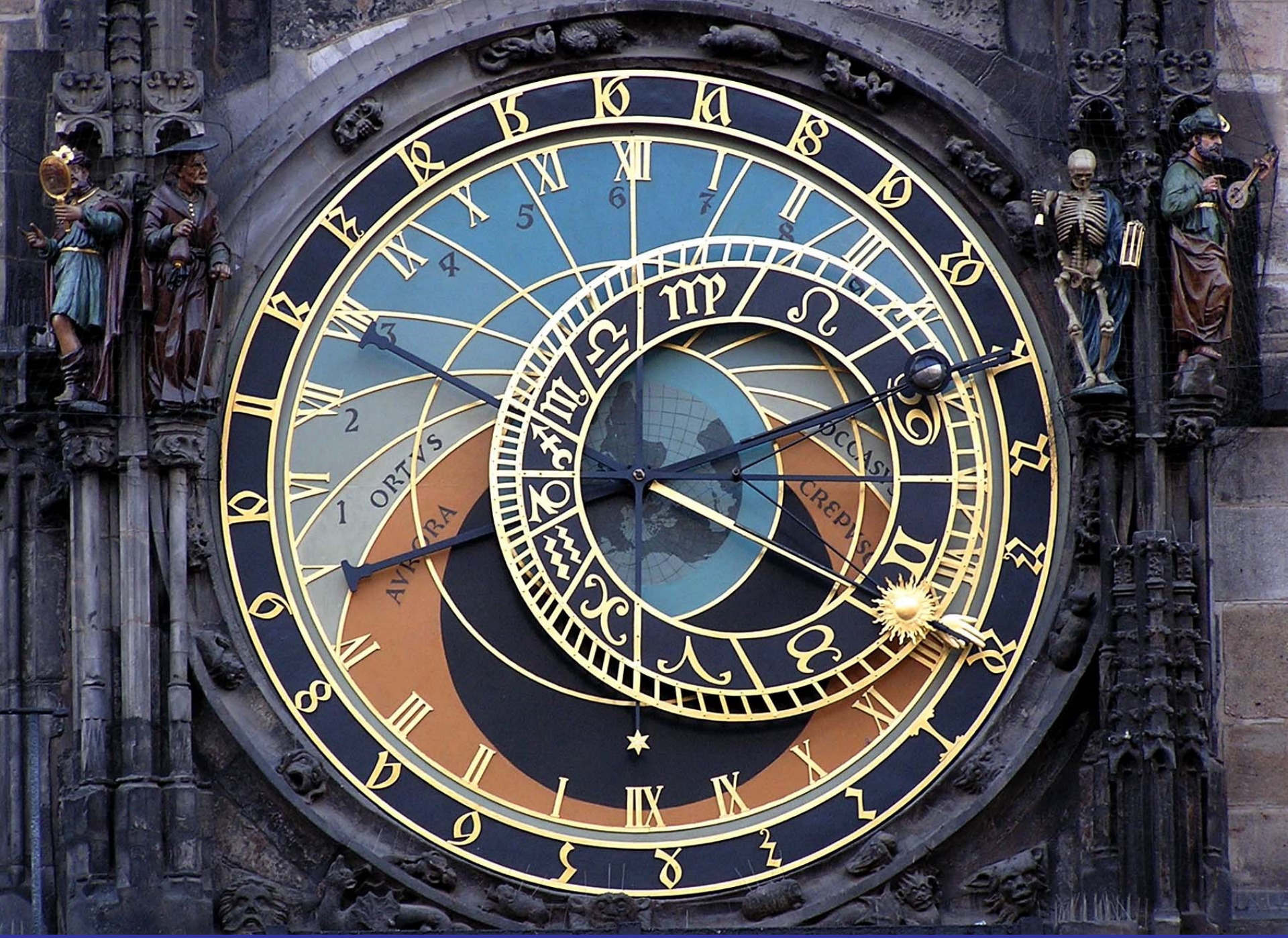
Europe

North Atlantic Ocean
North Sea
Baltic Sea
Irish Sea
English Channel
Bay of Biscay
Mediterranean Sea
Adriatic Sea
Tyrrhenian Sea
Ionian Sea
Aegean Sea
Black Sea
RUSSIA
ESTONIA
LATVIA
LITHUANIA
BELARUS
UKRAINE
MOLDOVA
POLAND
GERMANY
DENMARK
NETHERLANDS
BELGIUM
LUX.
FRANCE
SWITZ.
AUSTRIA
CZECH REPUBLIC
SLOVAKIA
HUNGARY
ROMANIA
BULGARIA
Serbia
BOSNIA AND HERZEGOVINA
CROATIA
SLOVENIA
ITALY
SPAIN
ANDORRA
PORTUGAL
IRELAND
UNITED KINGDOM
GREECE
ALB.
TURKE
Scale 1:19,500,000
Lambert Conformal Conic Projection,
standard parallels 40° N 56° N

Prague


PRAHA
MALOSTRANSKÁ
HRADČANY
MALÁ STRANA
STAROMĚSTSKÁ
JOSEFOV
ST. MĚSTO
NOVÉ MĚSTO
NÁMĚSTÍ REPUBLIKY
MŮSTEK
HLAVNÍ NÁDRAŽÍ
NÁRODNÍ TŘÍDA
MUZEUM
Pražský hrad
Belvedér
Karlův most
Mánesův most
Most Legií
Václavské náměstí
Wilsonova
Na příkopě
Revoluční
Klimentská
Na Poříčí
Celetná
Hybernská
Jindřišská
Vodičkova
Národní
Na Perštýně
Pařížská
Staroměstské nám.
Mariánské nám.
Staré zámecké schody
Letenská
Valdštejnská
Malostranské nám.
Velkopřevorské nám.
Maltézské nám.
Petřínské sady
Střelecký ostrov
Žofín
Národní divadlo
Masarykovo nádraží
Na Florenci
Petrské nám.
Haštalské nám.
Staré zámecké schody
Pražské Jezulátko
Nám. Kinských
Čelakovského sady

Staroměstské
náměstí
Astronomical
Clock
Malé
nám
U Radnice
Železná


ORTVS
AVRORA
CREPVS

Jan Ondřejův, called Šindel

- born in 1375 in Hradec Králové
- 1395 bachelor from Prague University
- 1399 master
- 1406 head of parish school of St. Nicholas
- 1410 rector of Charles University
- T. Brahe used his *Tabulae astronomicae*
- wrote several astronomical treatises
- asteroid No. 3847 is called **Šindel**

Theorem 1. *The stereographic projection is a conformal mapping.*

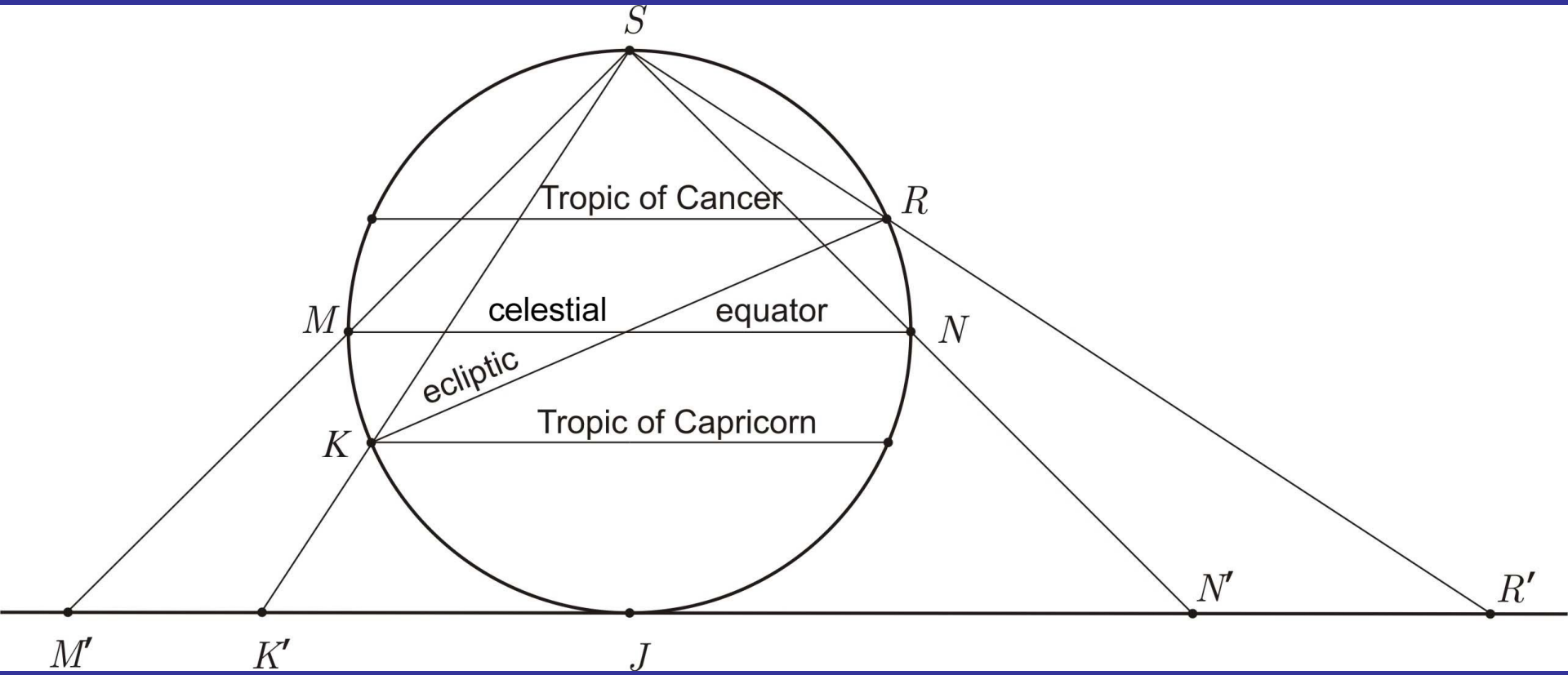
This property of the stereographic projection was already known to the Greek mathematician

Claudios Ptolemy (cca 90 – 160 A.D.)

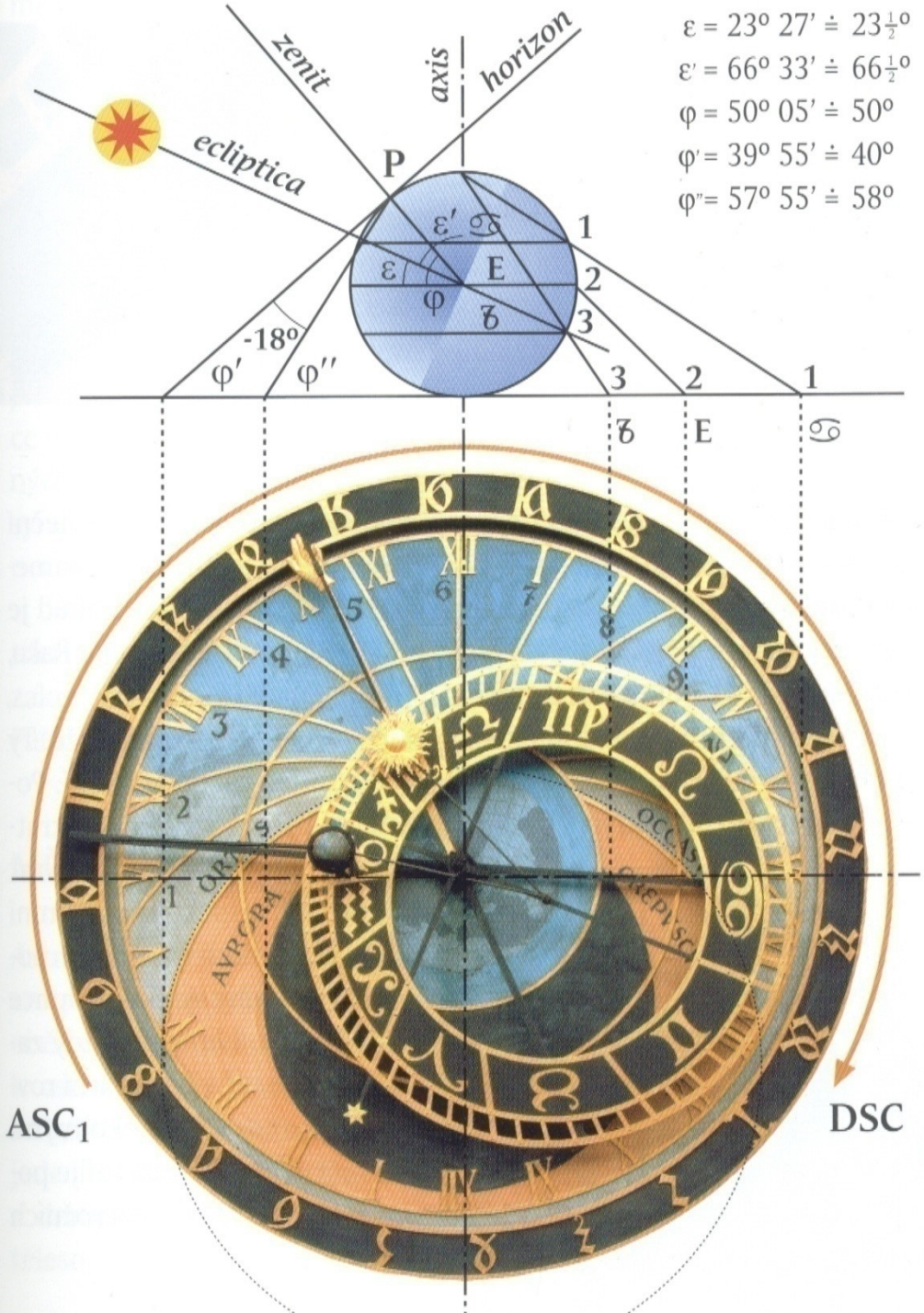
It is described in *Geographia* a *Planisphaerium.*

The angle between the ecliptics and the celestial equator is the same in the projection plane.

Dial as Astrolabium



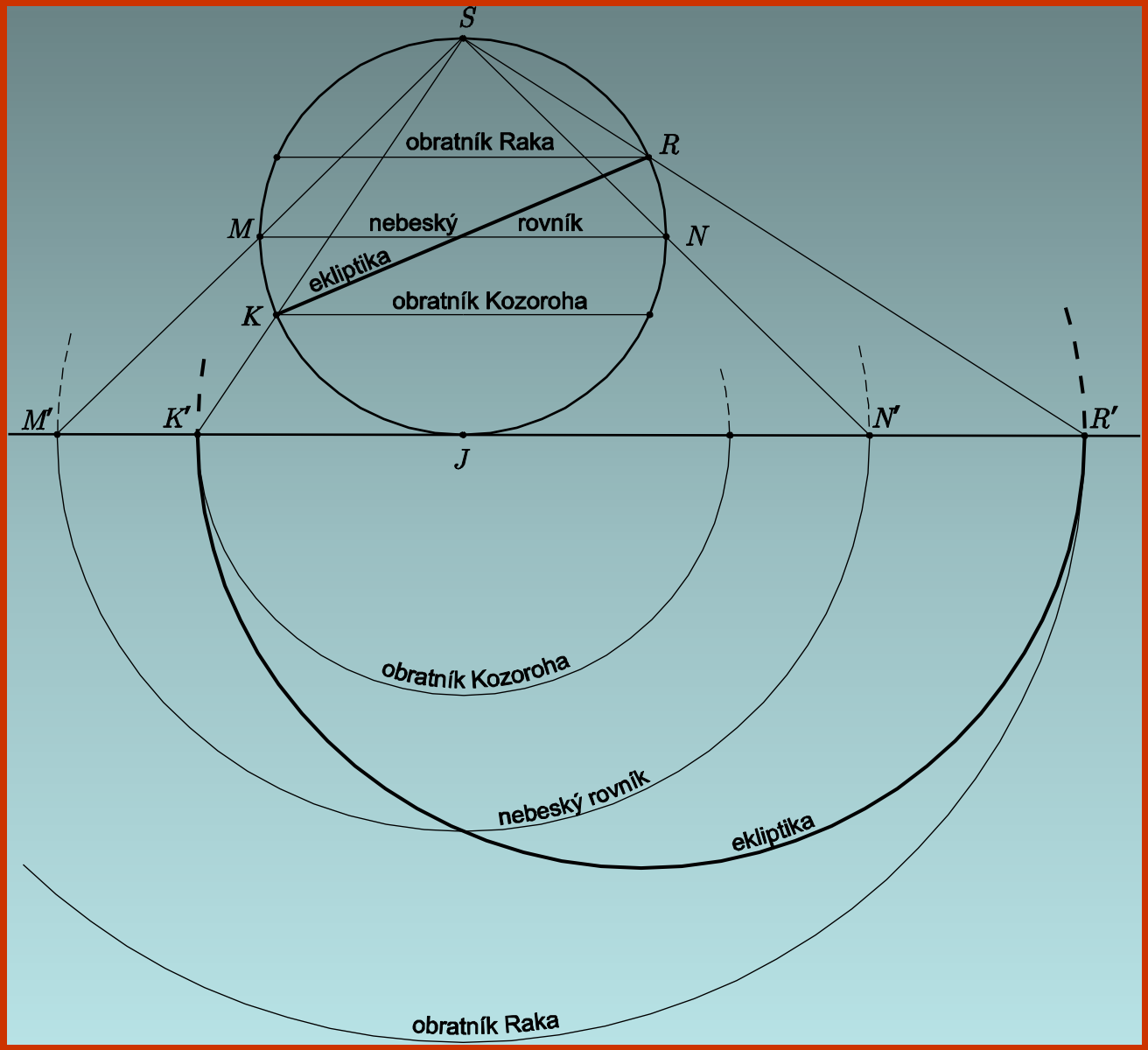
Stereographic projection of the celestial sphere


ε = 23º 27′ ≐ 23½º
ε′ = 66º 33′ ≐ 66½º
φ = 50º 05′ ≐ 50º
φ′ = 39º 55′ ≐ 40º
φ″ = 57º 55′ ≐ 58º
zenit
axis
horizon
ecliptica
P
E
-18º
φ′
φ″
1
2
3
ASC1
DSC

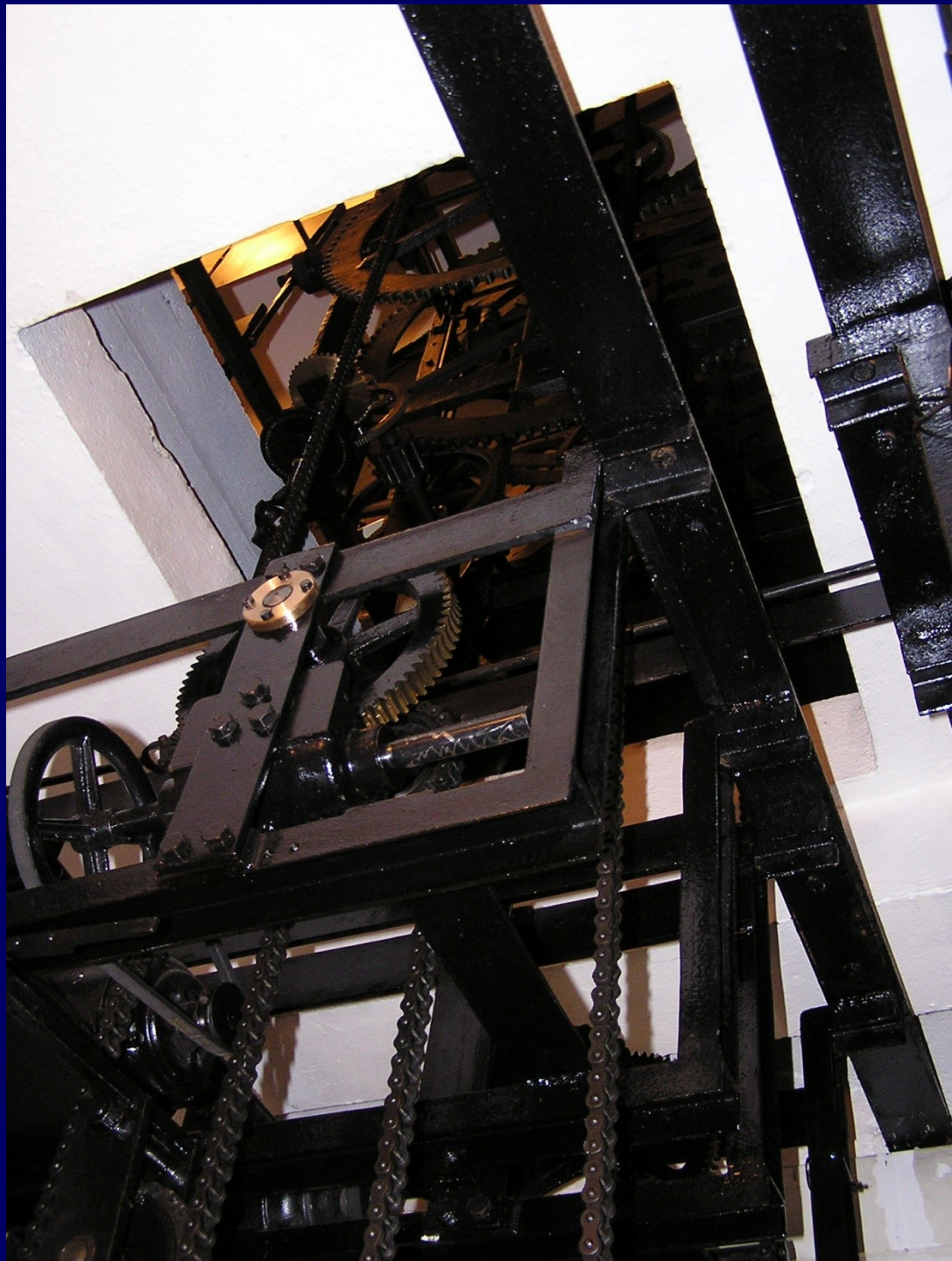
Theorem 2 (Ptolemy). *Any circle on the sphere which does not pass through the North Pole is mapped onto a circle as well.*

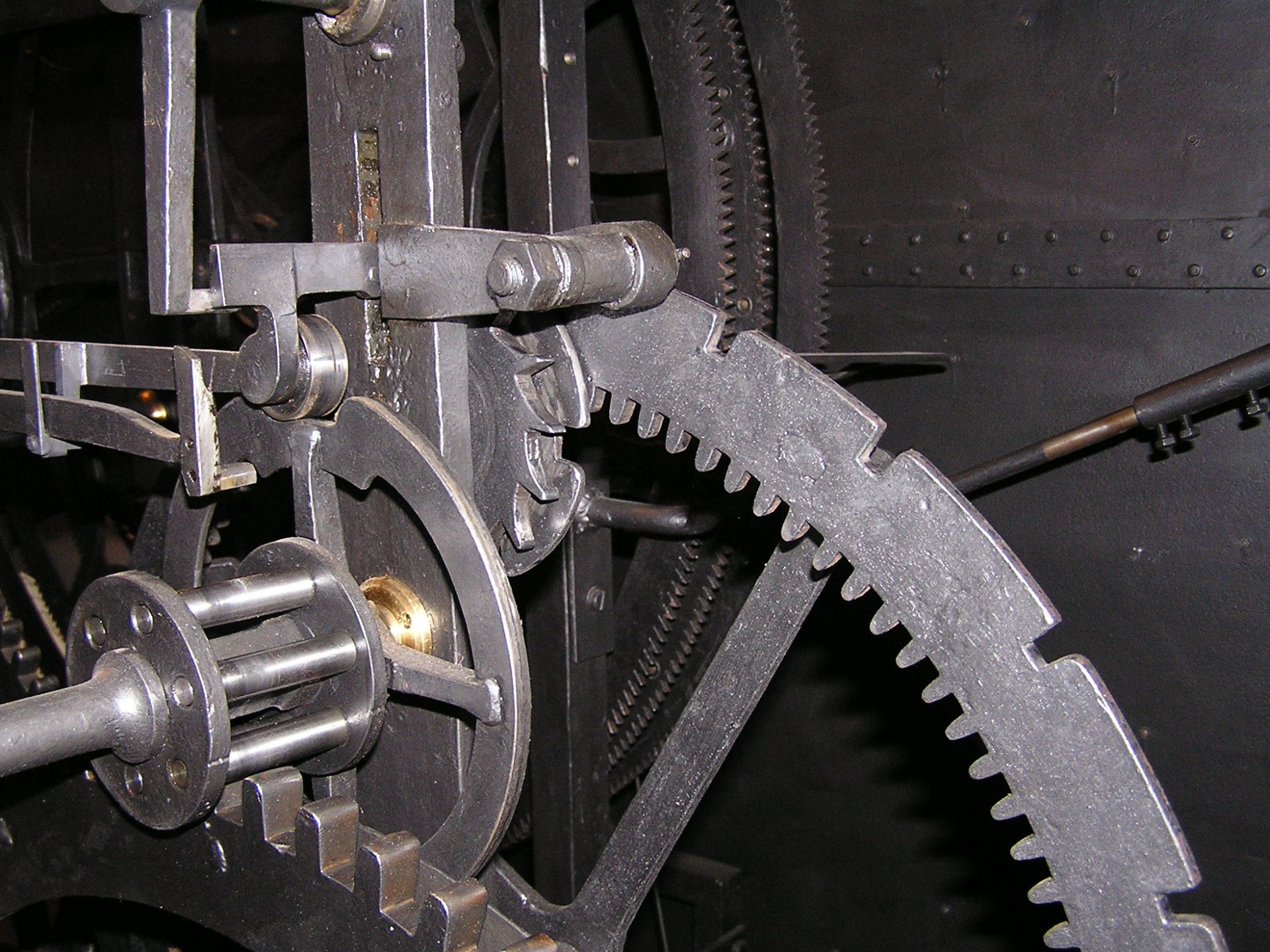
Theorem 3. *Any circle on the sphere passing through the North Pole is mapped onto a straight line.*

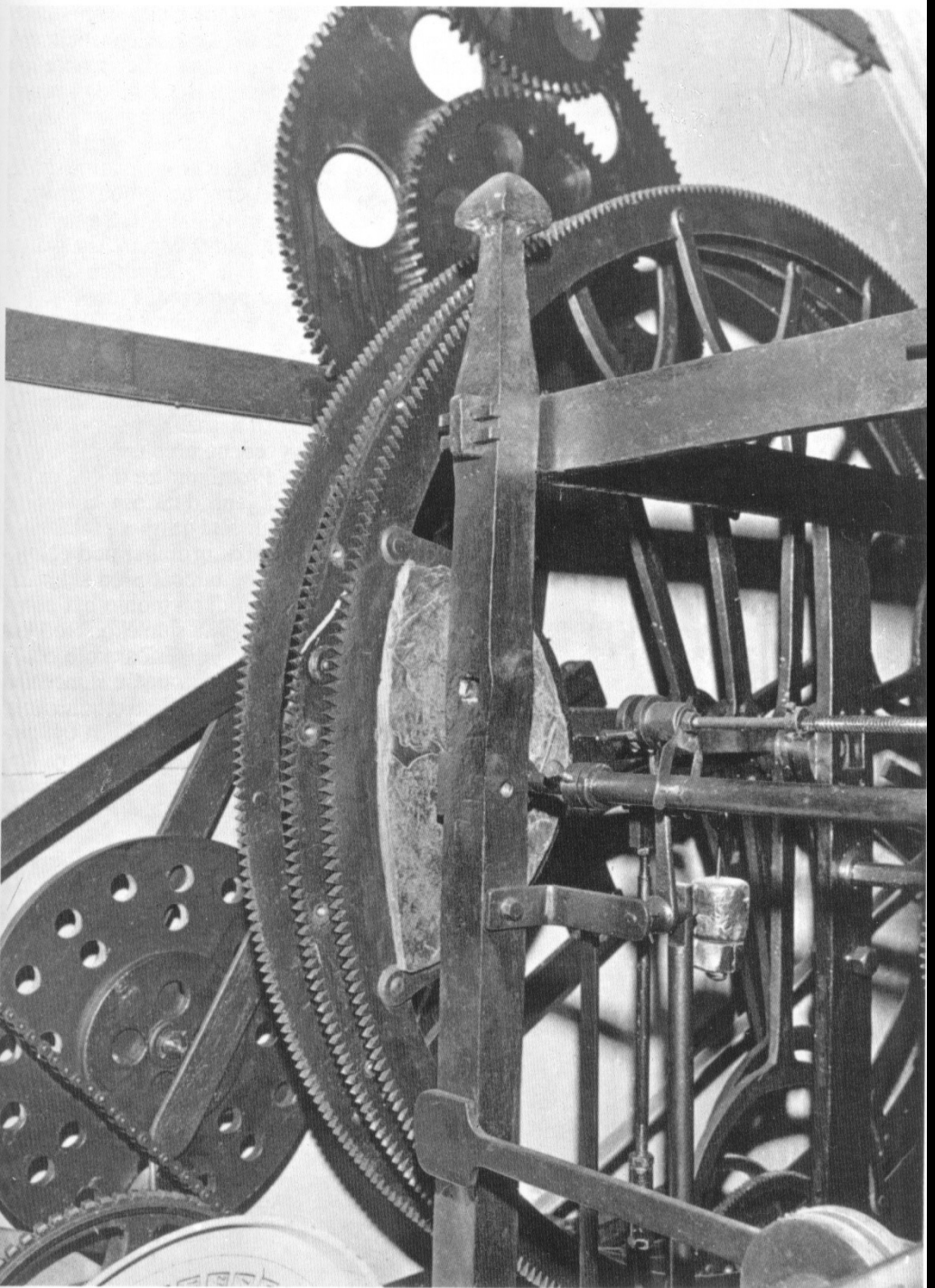
M. Křížek, A. Šolcová, J. Šolc: *Pražský orloj a stereografická projekce,* Matematika-fyzika-informatika, 2007.


S
obratník Raka
R
M
nebeský
rovník
N
ekliptika
K
obratník Kozoroha
M′
K′
N′
R′
J
obratník Kozoroha
nebeský rovník
ekliptika
obratník Raka

View of the main clockwork from the first to the second floor.







Three large original concentric gears of the same diameter which were originally driven on one axis by three pinions, each with 24 teeth.

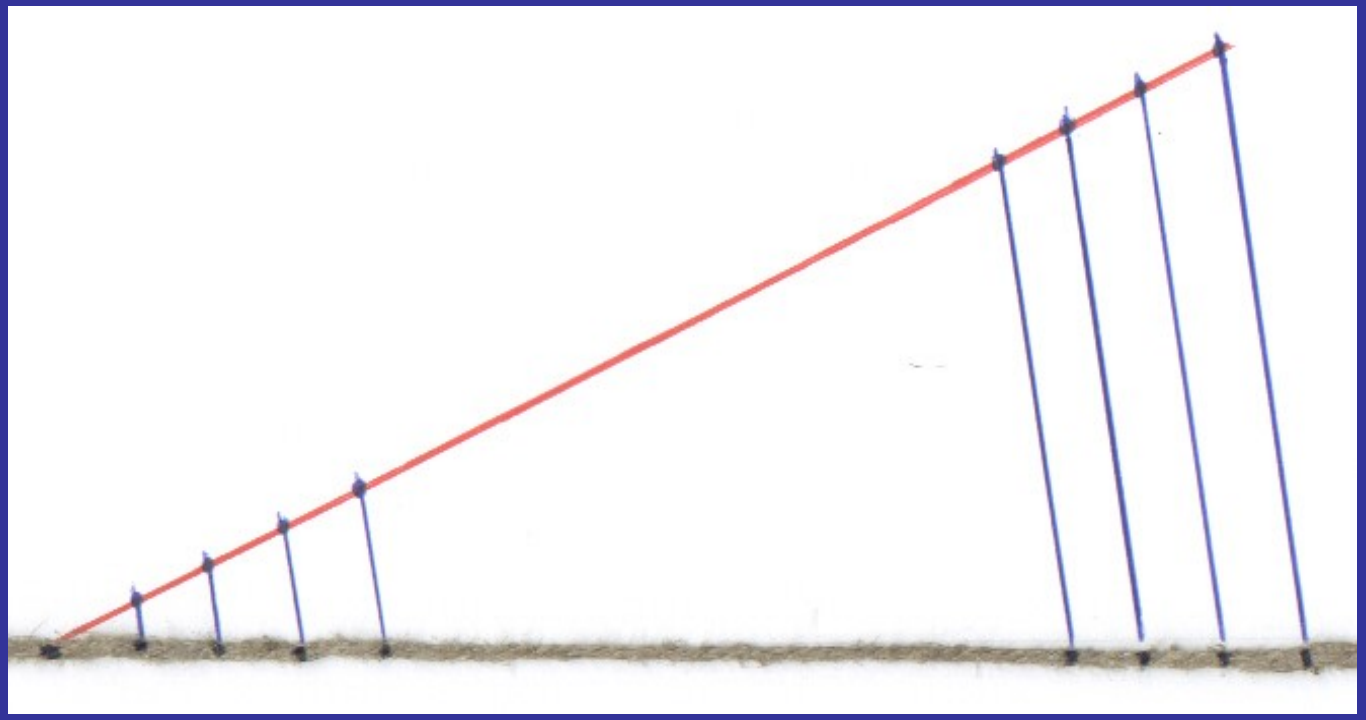
The first gear has 365 teeth and turns round the zodiac ring once per sidereal day (i.e., 23 h 56 min 4 s).

The second gear, which has 366 teeth, leads the solar pointer and turns round once per mean solar day (i.e, 24 h).

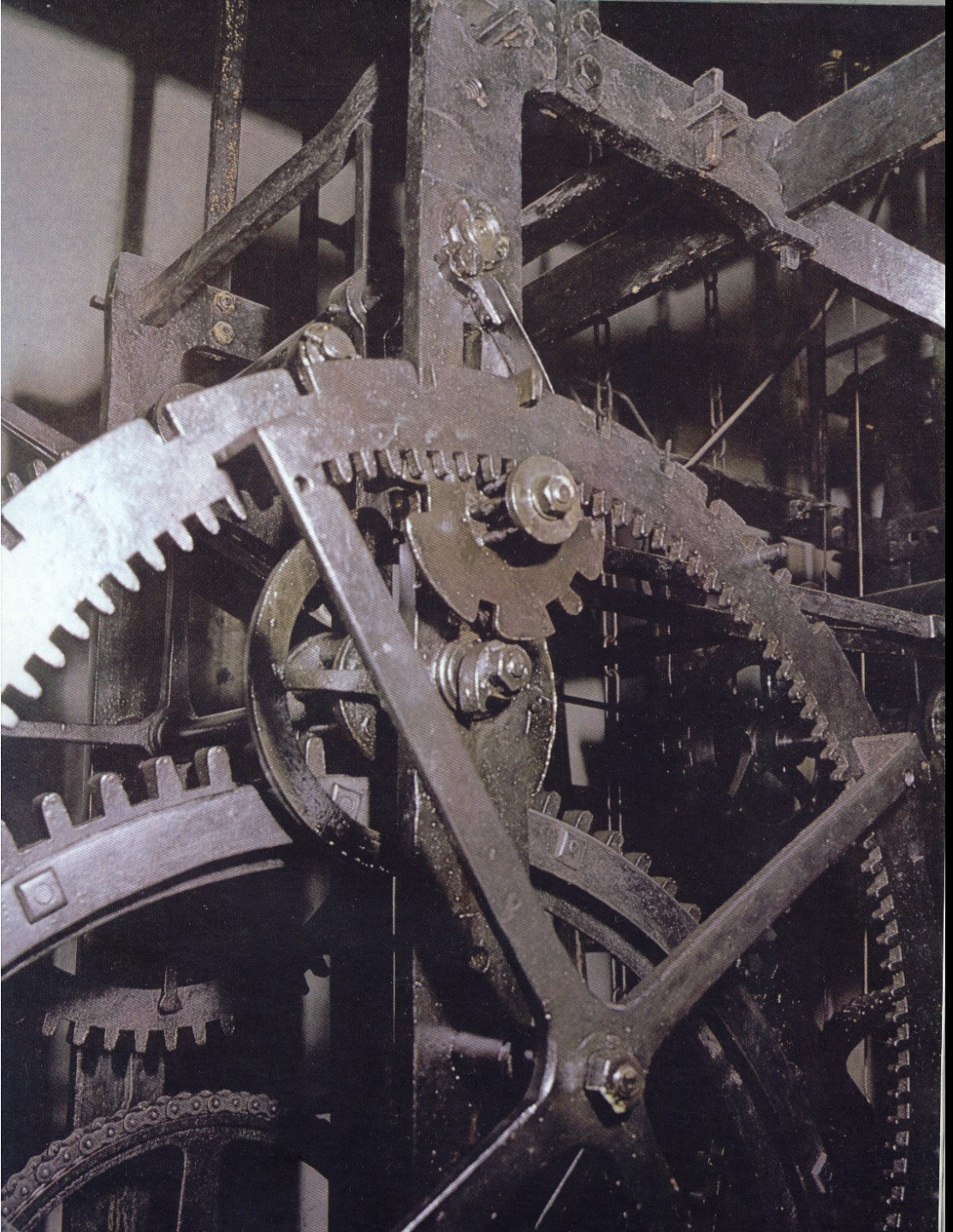
The third gear, which has 379 teeth, leads the Moon's hand and rotates according to the mean apparent motion of the Moon.

Let us ask the question, how could clockmakers in the beginning of the 15th century make such large gears with a relatively high accuracy?

Theorem 4 (Gauss). *Let p be an odd prime. Then there exists a Euclidean construction of the regular p-gon iff* $p = 2^n + 1$.



M. Křížek, F. Luca, L. Somer: *17 Lectures on Fermat numbers,* Springer, New York, 2001.



The bellworks contains a large gear with 24 slots at increasing distances. This arrangement allows for a periodic repetition of 1-24 strokes of the bell each day.

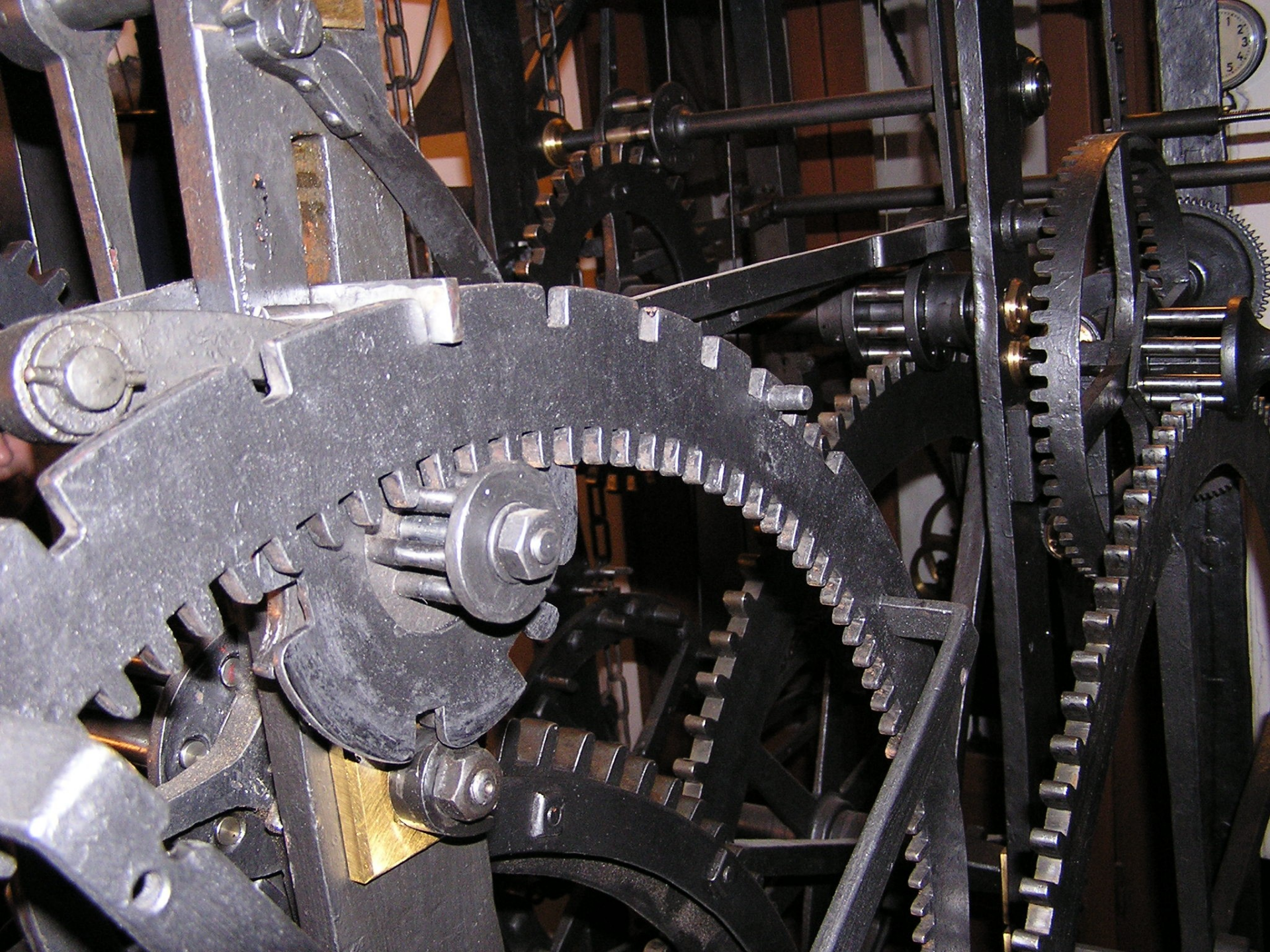
There is also a small gear whose circumference is divided by 6 slots into segments of arc lengths

$$1,\ 2,\ 3,\ 4,\ 3,\ 2\ .$$

These numbers form a period which repeats after each revolution and their sum is

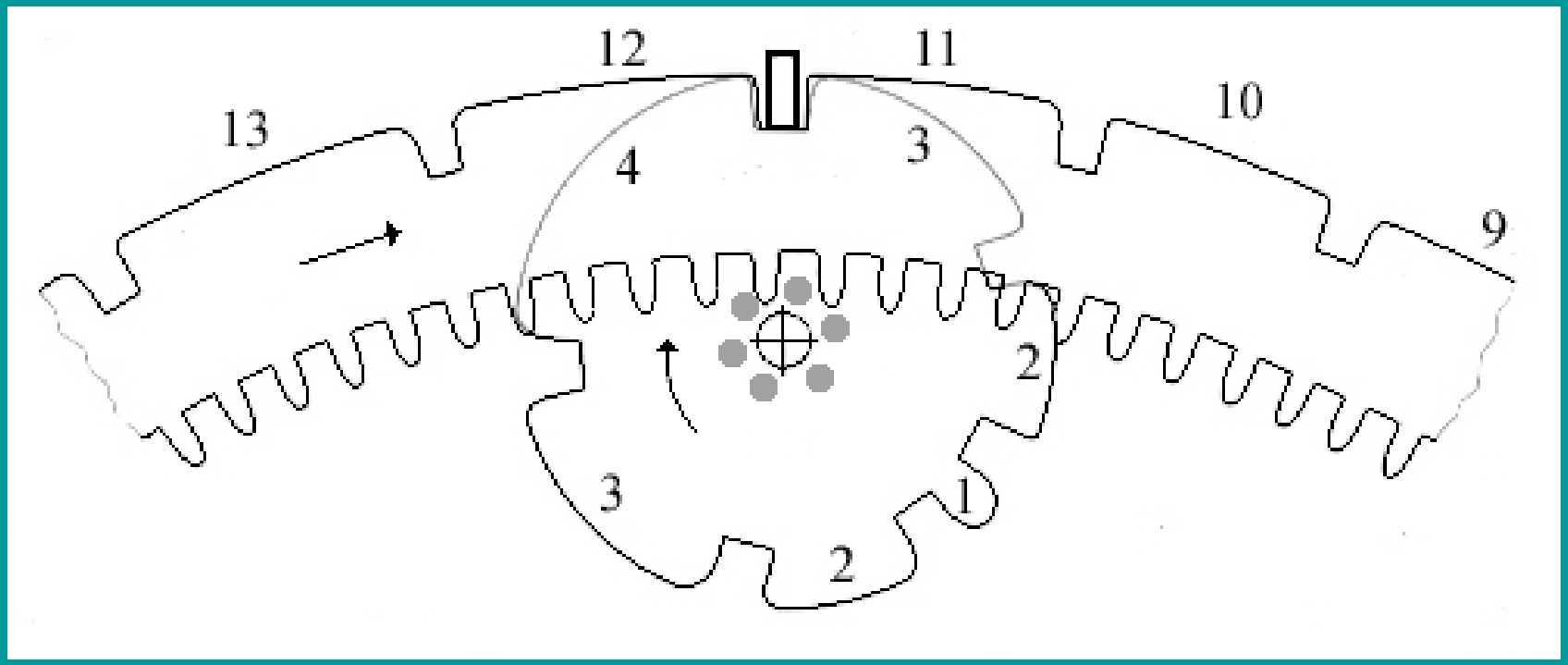
$$s = 15\ .$$

At the beginning of every hour a catch rises, both gears start to revolve and the bell chimes. The gears stop when the catch simultaneously falls down back into the slots on both gears.



$$1+2+3+\ldots+24=300$$

the number of bell strikes every day.
Since this number is divisible by *s*=15,
the small gear is always at the same position
at the beginning of each day.



Šindel sequence 123432

1 2 3 4 3 2 1 2 3 4 3 2 1 2 3 4 3 2

1 2 3 4 3 2 1 2 3 4 3 2 1 2

Further we show how *triangular numbers*

$$T_k = 1 + 2 + \ldots + k$$

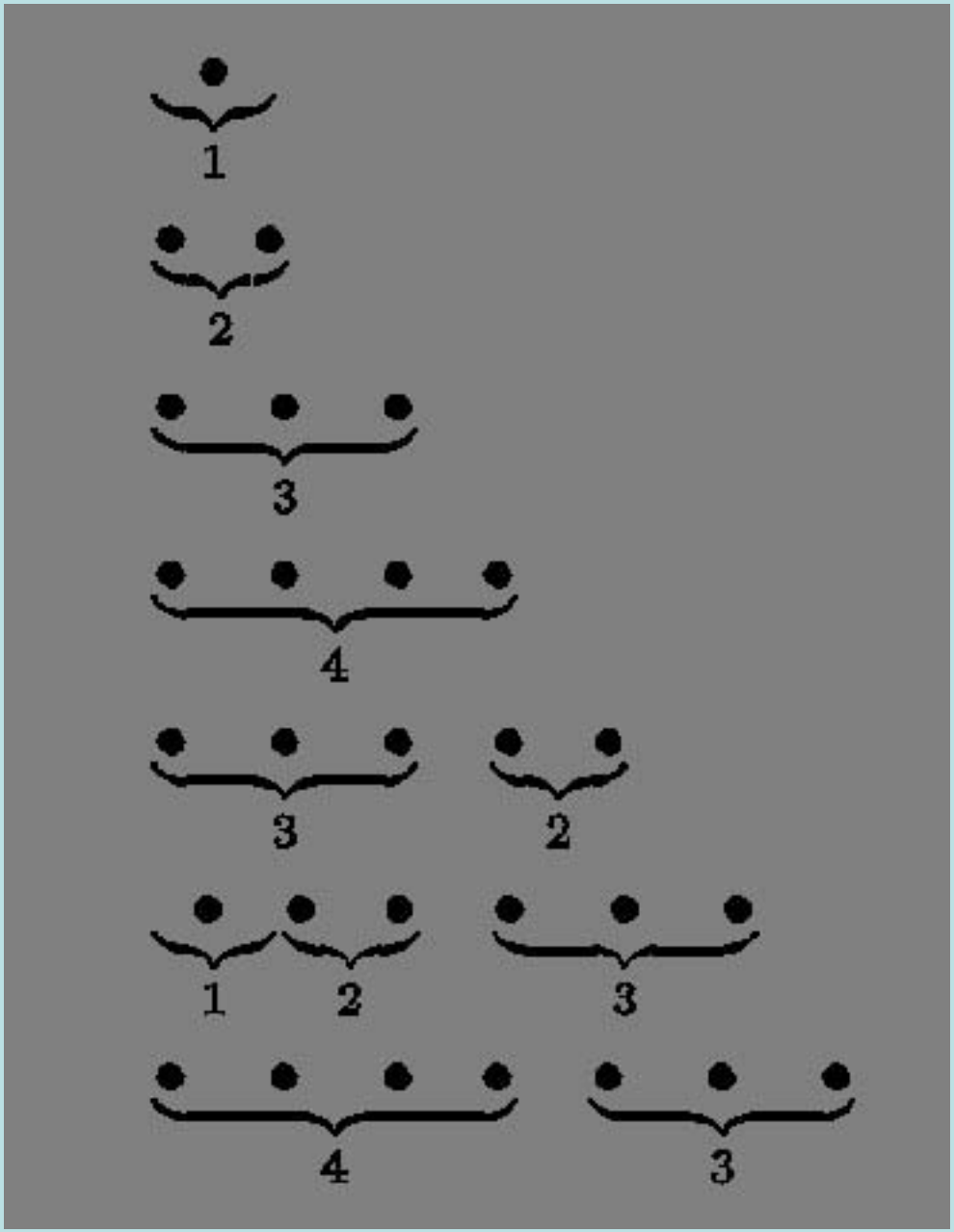
are connected with the astronomical clock.

For a periodic sequence $\{a_i\}$ we set

$$s = a_1 + a_2 + \ldots + a_p \ ,$$

where p is the period length.

T_7



A periodic sequence $\{a_i\}$ is said to be a *Šindel sequence*, if for any positive integer *k* there exists a positive integer *n* such that

$$T_k = a_1 + a_2 + \ldots + a_n$$

Theorem 5. *Let s be odd. A periodic sequence* $\{a_i\}$ *is a Šindel sequence, if for every k*<(s+1)/2 *there exists a positive integer n such that*

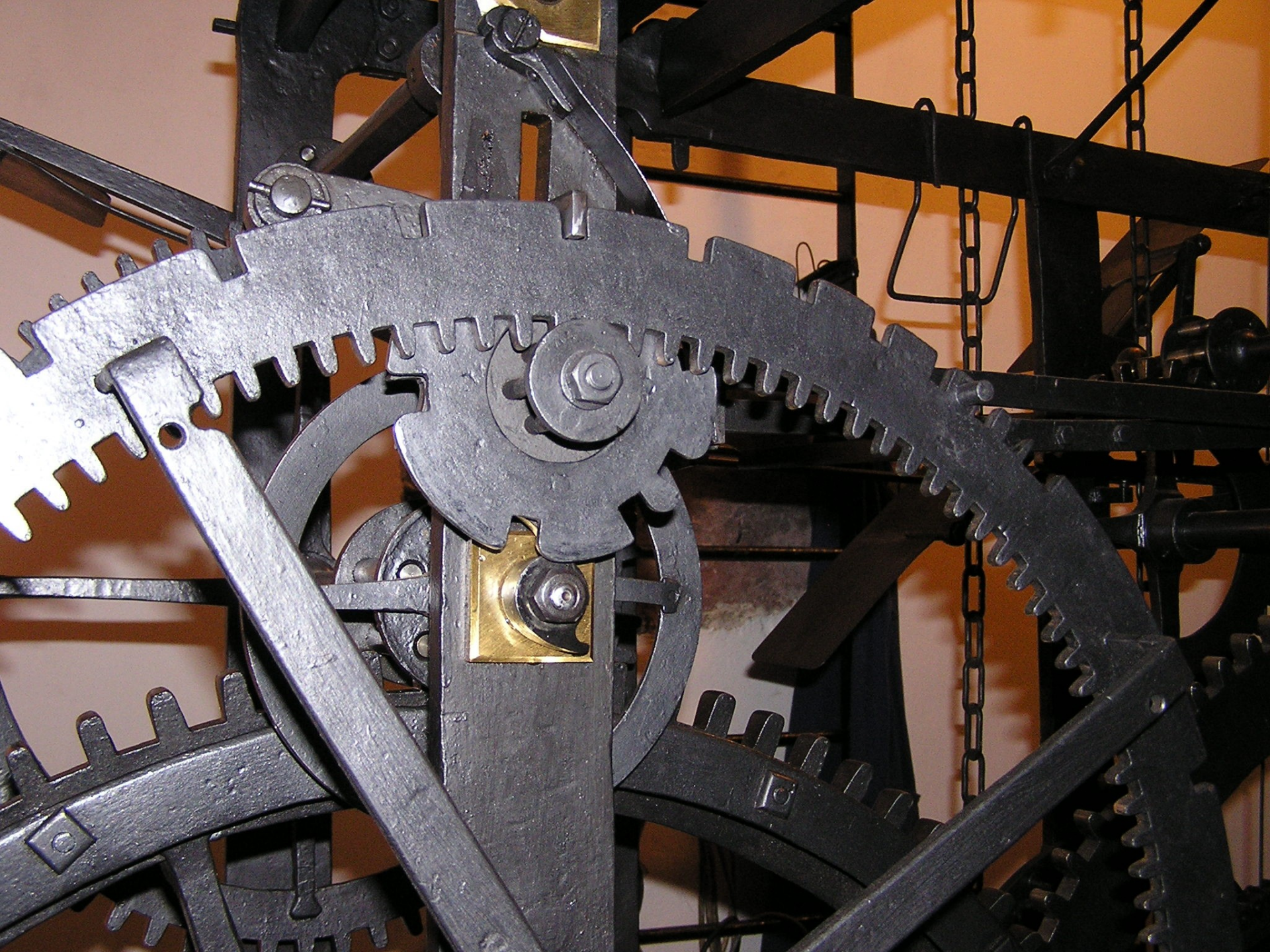
$$T_k = a_1 + a_2 + \ldots + a_n$$

Theorem 5a. *Let s be even. A periodic sequence* $\{a_i\}$ *is a Šindel sequence, if for any k<s there exists a positive integer n such that*

$$T_k = a_1 + a_2 + \ldots + a_n$$

Remark. Inequalities $k<(s+1)/2$ (resp. $k<s$) in Theorem 5 (resp. 5a) cannot be improved.

M. Křížek, L. Somer, A. Šolcová: *Jaká matematika se ukrývá v pražském orloji?,* Matematika-fyzika-informatika 16 (2006/ 2007), 129-137.



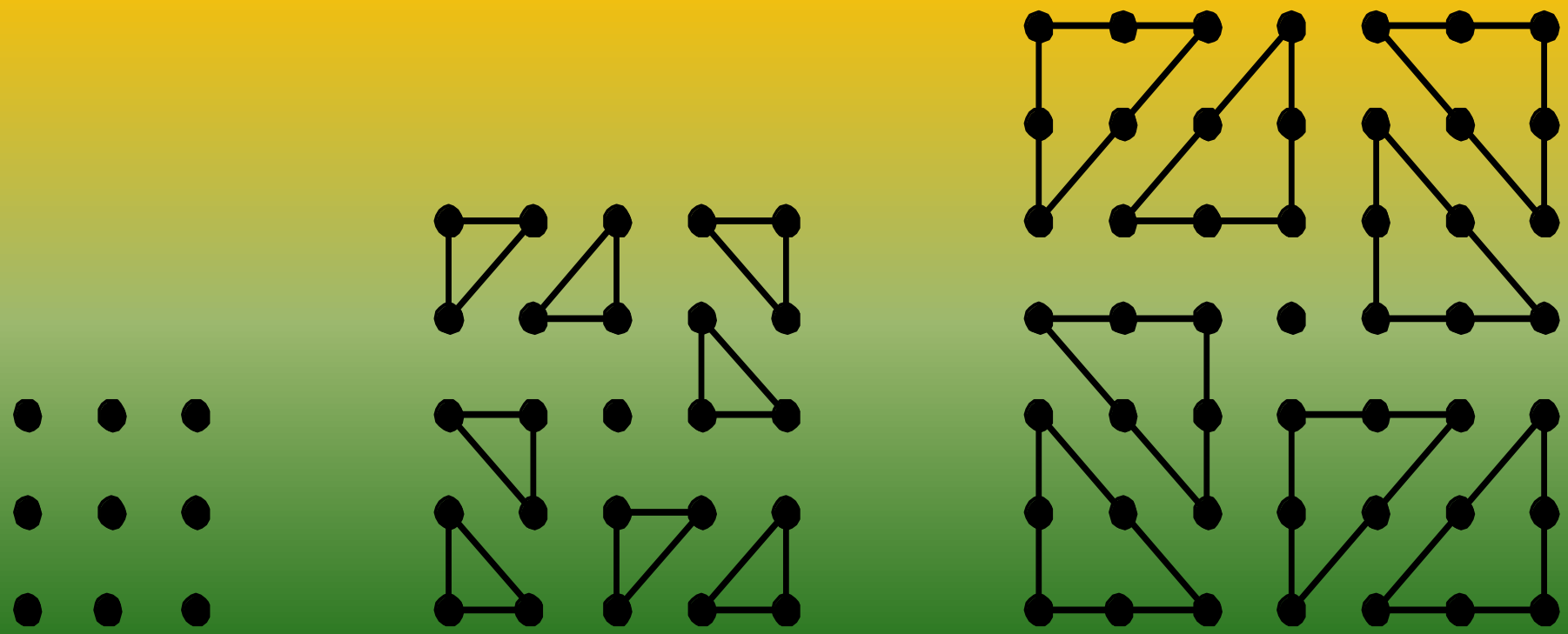
Theorem 6. *A periodic sequence* $\{a_i\}$ *is a Šindel sequence if and only if for every* $n = 1, 2, \ldots, p$ *and every* $j = 1, 2, \ldots, a_n$ *the number*

$$8(a_1 + \ldots + a_n - j) + 1$$

is a quadratic nonreziduum modulo s.

Theorem 7 (Plutarchos).
A positive integer r *is a triangular number if and only if* $8r + 1$ *is a square.*

M. Křížek, A. Šolcová, L. Somer:
What mathematics is hidden behind the astronomical clock of Prague,
Proc. of the IAU XXVI.General Assembly, Cambridge Univ. Press, 2007.



Definition.

A Šindel sequence $\{b_i\}$ with the minimal period $p + 1$ is said to be *composite,* if there exists a Šindel sequence $\{a_i\}$ and a positive integer k such that

$$a_i = b_i \qquad \text{for } i = 1, \ldots, k-1$$

$$a_k = b_k + b_{k+1}$$

$$a_i = b_i \qquad \text{for } i = k, \ldots, p$$

Example. The period 1, 2, 3, 2, 2, 3, 2 derived from the period 1, 2, 3, 4, 3, 2 produces a composite Šindel sequence.

Definition. A Šindel sequence is called *primitive* if it is not composite.

Theorem 8. *For any positive integer s there exists a unique primitive Šindel sequence.*

The sequence $1, 1, \ldots, 1$ is called a *trivial* Šindel sequence.

Theorem 9. *A primitive Šindel sequence is trivial if and only if s is a power of 2.*

M. Křížek, A. Šolcová, L. Somer: *Construction of Šindel sequences,* Comment. Math. Univ. Carolin. 48 (2007), 1-16.

Primitive Šindel sequences

s	
1	1
2	1 1
3	1 2
4	1 1 1 1
5	1 2 2
6	1 2 1 2
7	1 2 3 1
8	1 1 1 1 1 1 1 1
9	1 2 3 3
10	1 2 2 1 2 2
11	1 2 1 2 4 1
12	1 2 1 2 1 2 1 2
13	1 1 1 3 2 2 3
14	1 2 3 1 1 2 3 1
15	**1 2 3 4 3 2**
16	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
17	1 1 1 1 2 4 1 4 2
18	1 2 3 3 1 2 3 3
19	1 1 1 3 1 2 1 5 2 2
20	1 2 2 1 2 2 1 2 2 1 2 2

Theorem 10. *For any positive integer k there exists m>k and a Šindel sequence* $\{a_i\}$ *so that*

$$a_m = k.$$

Example. The period

1, 2, 3, 4, 5, 3, 3, 7, 2, 3, 3, 9

for $p = 12$ and $s = 45$ yields a Šindel sequence with a relatively large element 9 .

M. Křížek, A. Šolcová, L. Somer: *Construction of the Šindel sequences,* Proc. Internat. Conf. PANM 13, Prague, 2006, 156-164.

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Šindel sequence for $s = 365$

1 2 3 4 1 2 2 3 3 4 3 5 3 5 4 1 4 5

3 5 3 4 6 2 3 5 4 1 9 1 4 1 **14** **10** 1 5

15 2 3 2 3 2 8 2 **17** 1 5 5 9 6 4 5 1 4

1 5 4 6 7 2 5 5 1 **10** 2 7 8 5 2 1 4 5

3 **12** **10** **13** 5 2 7

1793

v roce 1793. Část kolorovaného leptu Caspara Plutha podle Filipa a Františka Hegra. Pozoruhodné jsou postavičky v otevřených okénkách

1794

The height
of the townhall tower
is 59 m = 100 ells.
It was founded
in the first half
of the 14th century.

[70] Filip a František Hegerové: *Staroměstské náměstí, 1794*

May 1945