What mathematics is hidden behind the astronomical clock of Prague?

Construction of Šindel sequences

Michal Křížek
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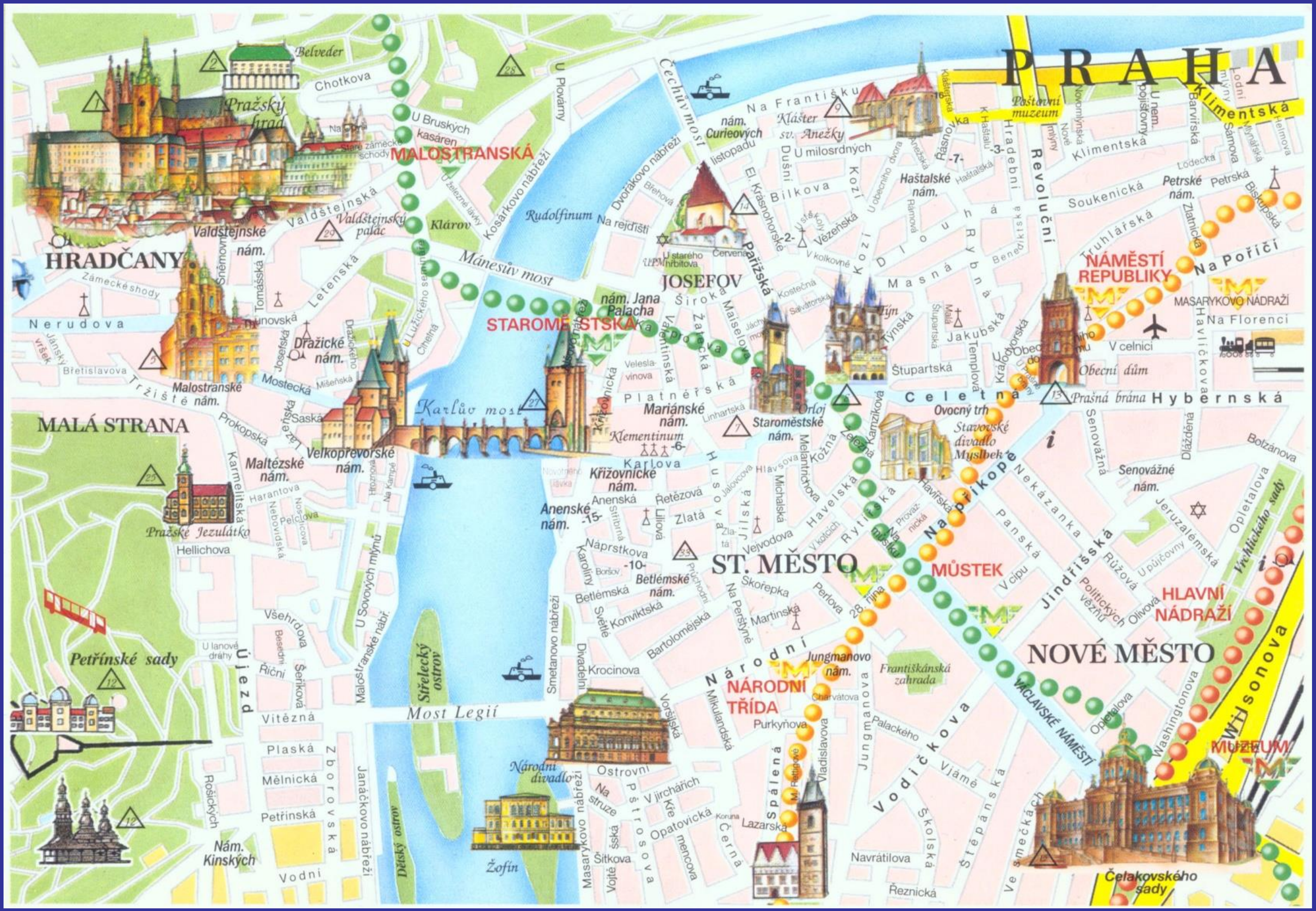
Europe

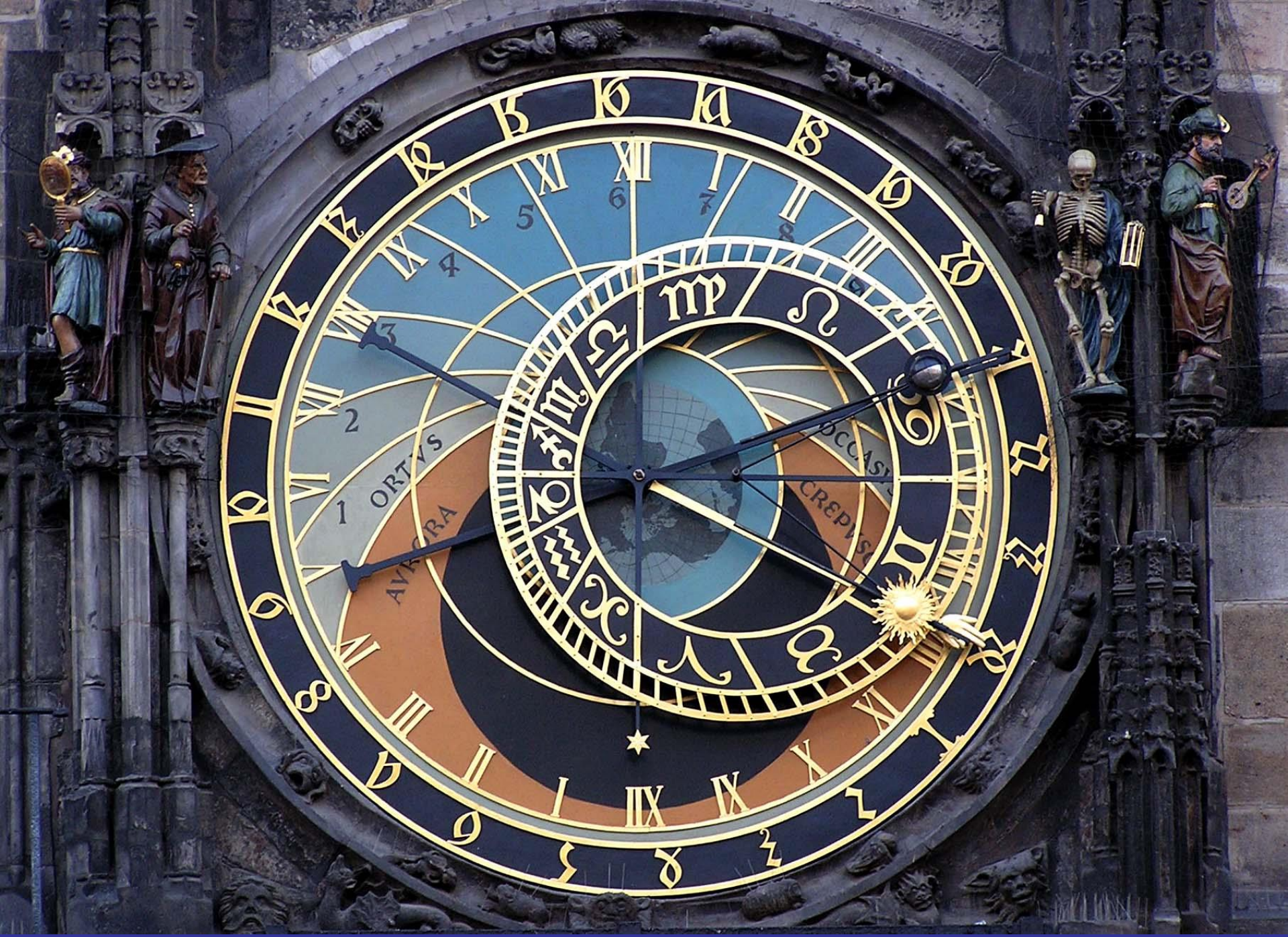
North Atlantic Ocean
North Sea
Baltic Sea
Irish Sea
English Channel
Bay of Biscay
Mediterranean Sea
Adriatic Sea
Tyrrhenian Sea
Ionian Sea
Black Sea
Aegean Sea
Rockall (U.K.)
Hebrides
Orkney Islands
Aberdeen
Edinburgh
Newcastle
Belfast
UNITED KINGDOM
Dublin
IRELAND
Liverpool
Cardiff
London
Amsterdam
NETHERLANDS
Brussels
BELGIUM
LUX.
Luxembourg
Paris
Le Havre
Nantes
Bordeaux
FRANCE
Bilbao
Porto
PORTUGAL
SPAIN
Madrid
Barcelona
ANDORRA
Valencia
Sevilla
Málaga
Lisbon
Algiers
Marseille
Corsica
Sardinia
Balearic Islands
Tunis
Valletta
Palermo
Sicily
Naples
Rome
ITALY
Florence
Genoa
Turin
Milan
Venice
Bern
SWITZ.
Geneva
Lyon
Strasbourg
Stuttgart
Munich
Frankfurt
Bonn
Hannover
Hamburg
Berlin
Leipzig
GERMANY
Copenhagen
DENMARK
Oslo
Bergen
Stockholm
Göteborg
Gävle
Aland Islands
Gotland
Oland
Bornholm
Kaliningrad
RUSSIA
Rostock
Gdańsk
Poznań
POLAND
Warsaw
Wrocław
Kraków
Prague
CZECH REPUBLIC
SLOVAKIA
Bratislava
Vienna
AUSTRIA
Graz
Budapest
HUNGARY
Pécs
Ljubljana
SLOVENIA
Zagreb
CROATIA
BOSNIA AND HERZEGOVINA
Sarajevo
Belgrade
Serbia
Podgorica
Montenegro
Tirane
ALB.
Skopje
Macedonia
Sofia
BULGARIA
Varna
Constanta
Bucharest
ROMANIA
Cluj-Napoca
MOLDOVA
Chişinău
Odesa
UKRAINE
L'viv
Kiev
Brest
Minsk
BELARUS
Smolensk
Riga
LATVIA
Vilnius
LITHUANIA
Tallinn
ESTONIA
RUSSIA
Istanbul
Thessaloniki
Ioannina
Corfu
GREECE
Athens
Peloponnisos
Crete
Rhodes
Scale 1:19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N 56°N

Prague



PRAHA
HRADČANY
MALÁ STRANA
JOSEFOV
ST. MĚSTO
NOVÉ MĚSTO
Pražský hrad
Karlův most
Mánesův most
Most Legií
Petřínské sady
MALOSTRANSKÁ
STAROMĚSTSKÁ
NÁMĚSTÍ REPUBLIKY
MŮSTEK
NÁRODNÍ TŘÍDA
HLAVNÍ NÁDRAŽÍ
MUZEUM
Václavské náměstí
Na příkopě
Národní
Wilsonova
Staroměstské nám.
Malostranské nám.
Nerudova
Celetná
Hybernská

Staroměstské
náměstí
Malé
nám
Astronomical
Clock
Železná


1 ORTVS
AVRORA

Jan Ondřejův, called Šindel

- born in 1375 in Hradec Králové
- 1395 bachelor from Prague University
- 1399 master
- 1406 head of parish school of St. Nicholas
- 1410 rector of Charles University
- T. Brahe used his *Tabulae astronomicae*
- wrote several astronomical treatises
- asteroid No. 3847 is called **Šindel**

Theorem 1. *The stereographic projection is a conformal mapping.*

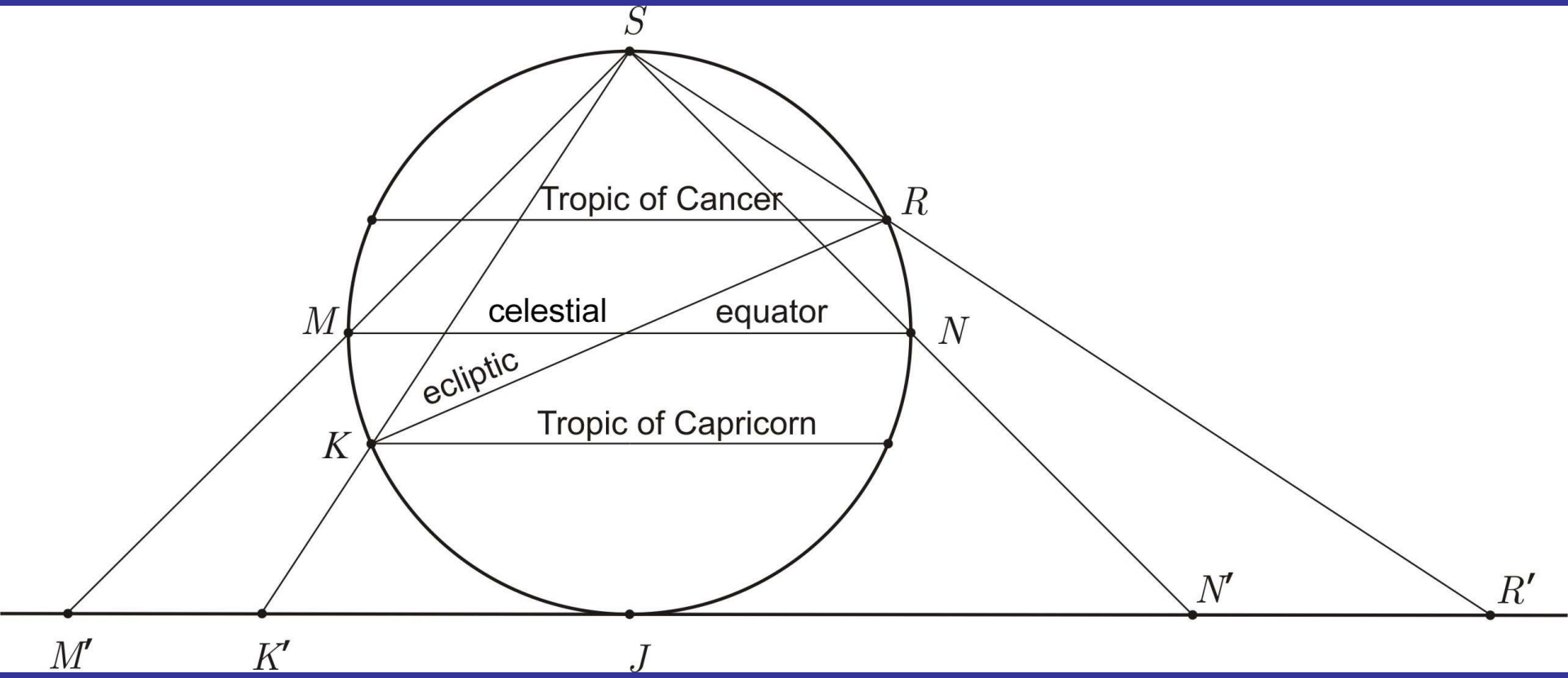
This property of the stereographic projection was already known to the Greek mathematician

Claudios Ptolemy (cca 90 – 160 A.D.)

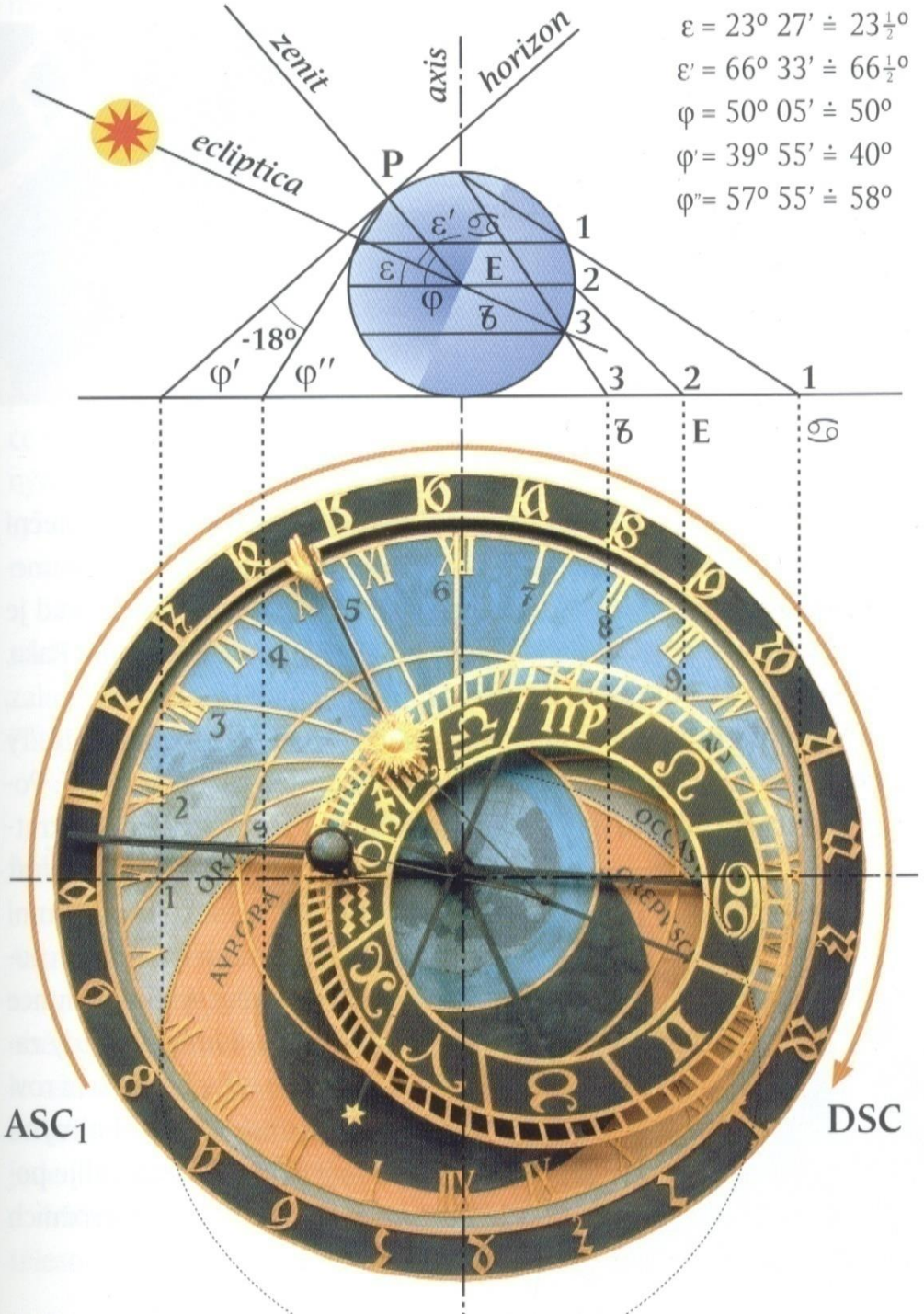
It is described in *Geographia* a *Planisphaerium.*

The angle between the ecliptics and the celestial equator is the same in the projection plane.

Dial as Astrolabium



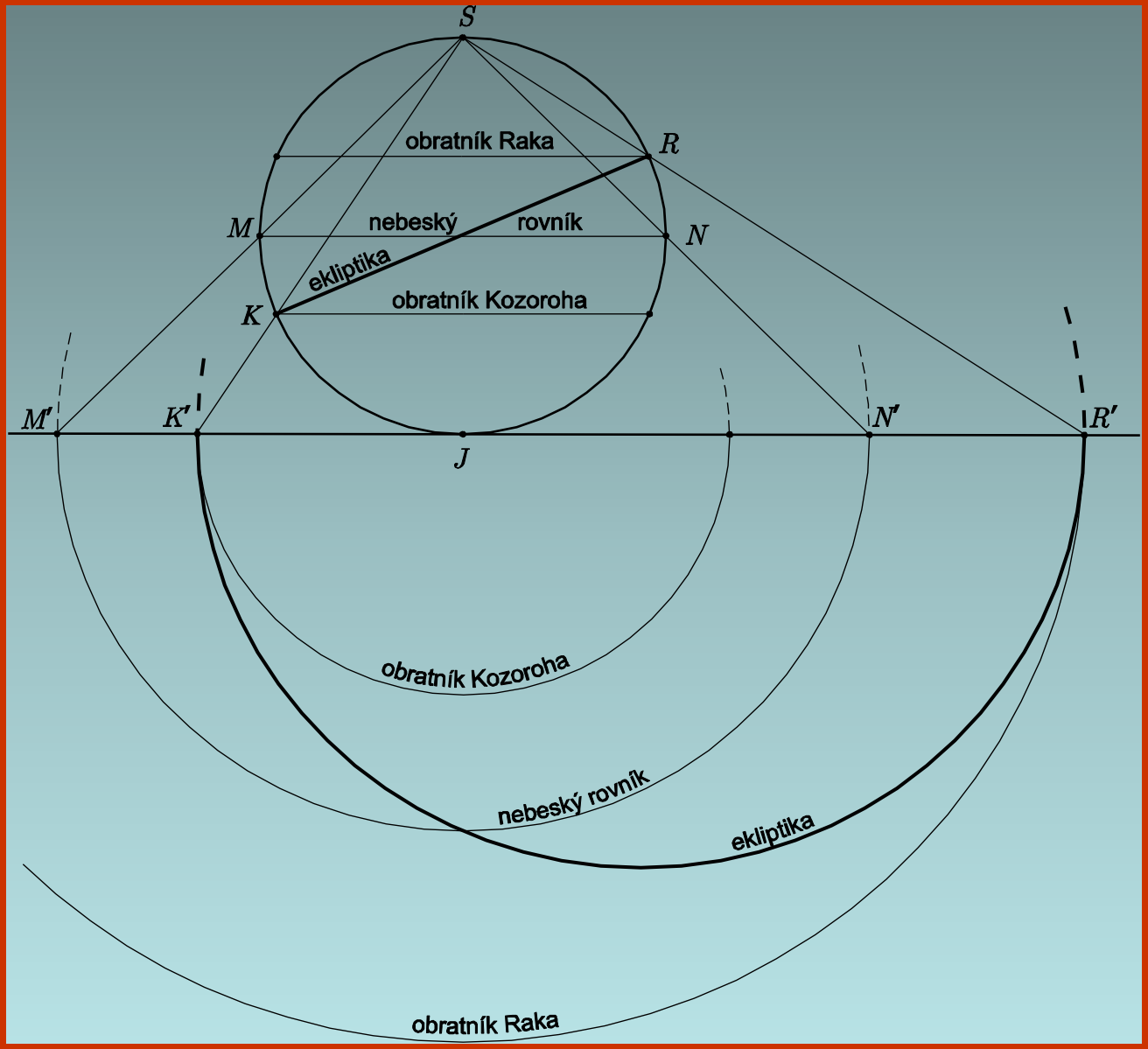
Stereographic projection of the celestial sphere


ε = 23° 27′ ≐ 23½°
ε′ = 66° 33′ ≐ 66½°
φ = 50° 05′ ≐ 50°
φ′ = 39° 55′ ≐ 40°
φ″ = 57° 55′ ≐ 58°
zenit
axis
horizon
ecliptica
P
ε′
ε
E
φ
1
2
3
-18°
φ′
φ″
3
2
1
E
ASC1
DSC

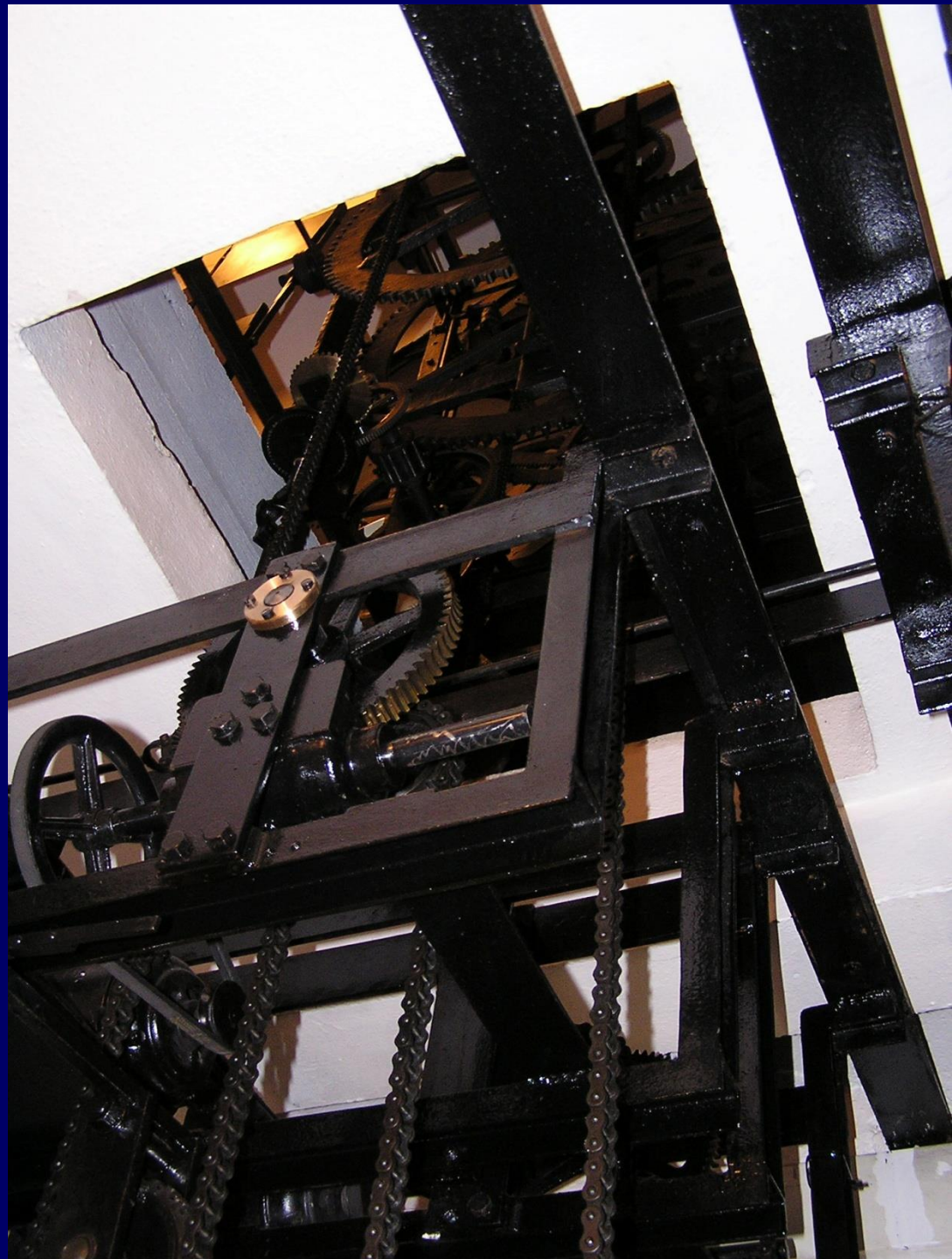
Theorem 2 (Ptolemy). *Any circle on the sphere which does not pass through the North Pole is mapped onto a circle as well.*

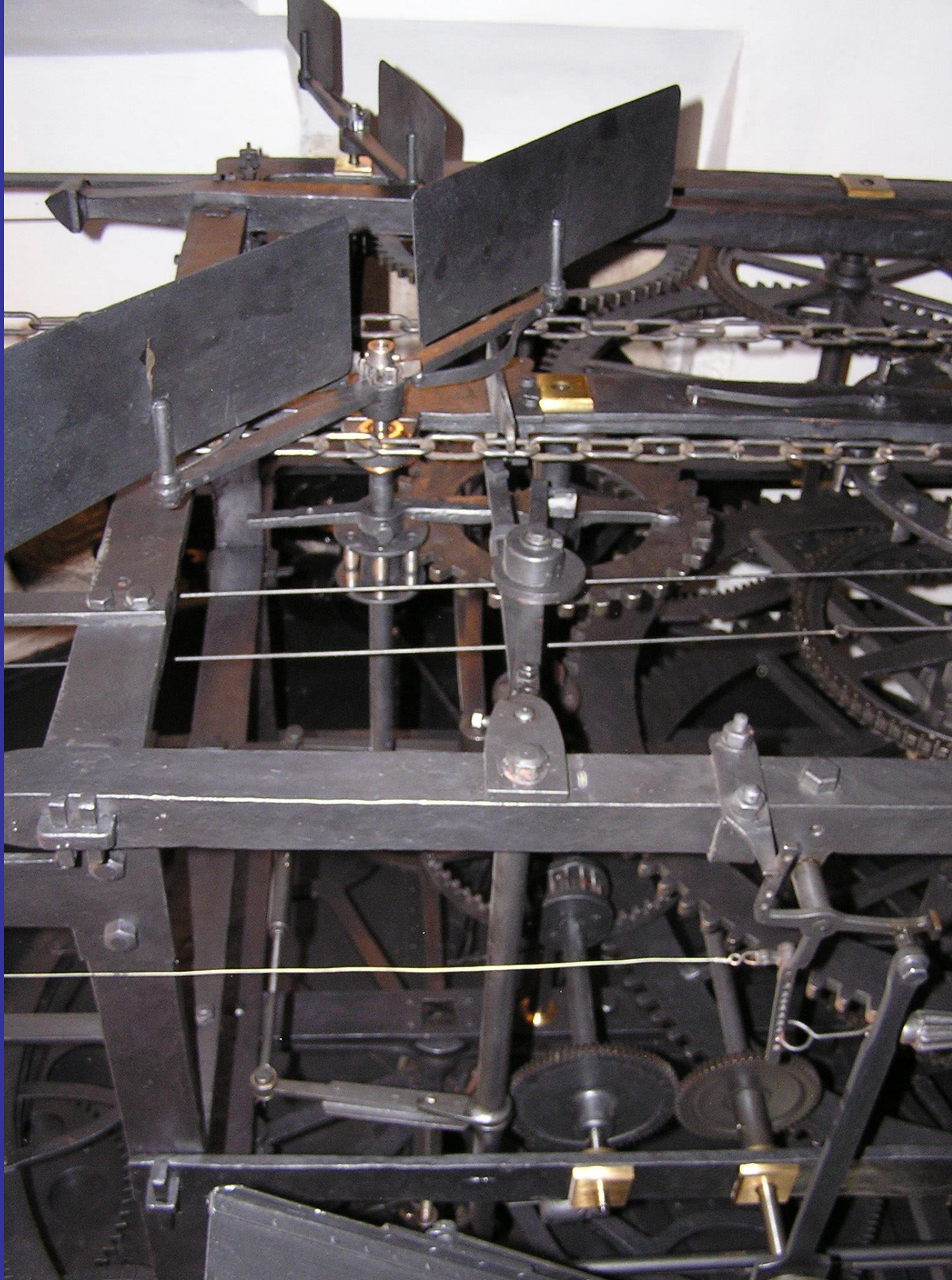
Theorem 3. *Any circle on the sphere passing through the North Pole is mapped onto a straight line.*

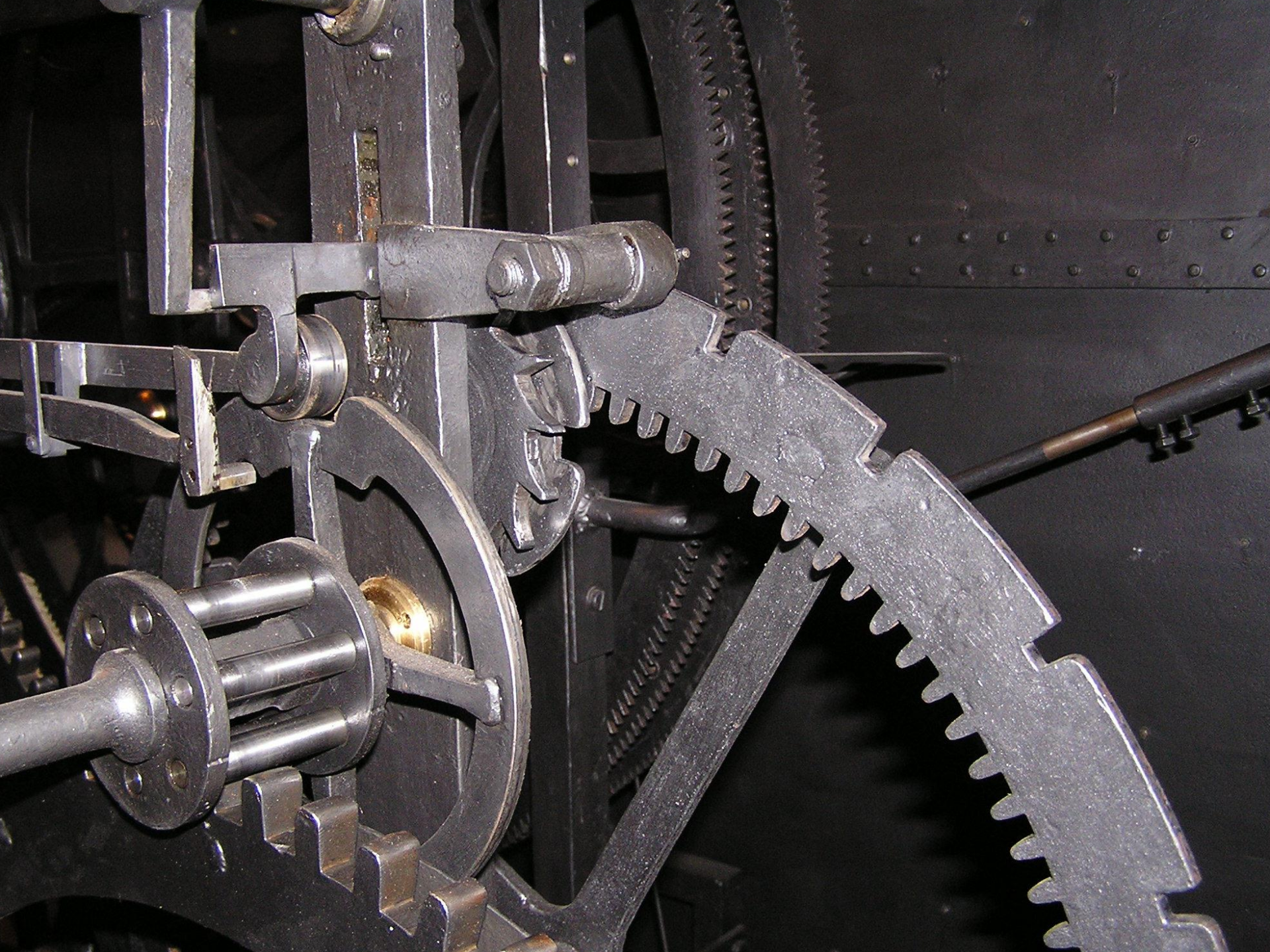
M. Křížek, A. Šolcová, J. Šolc: *Pražský orloj a stereografická projekce,* Matematika-fyzika-informatika, 2007.

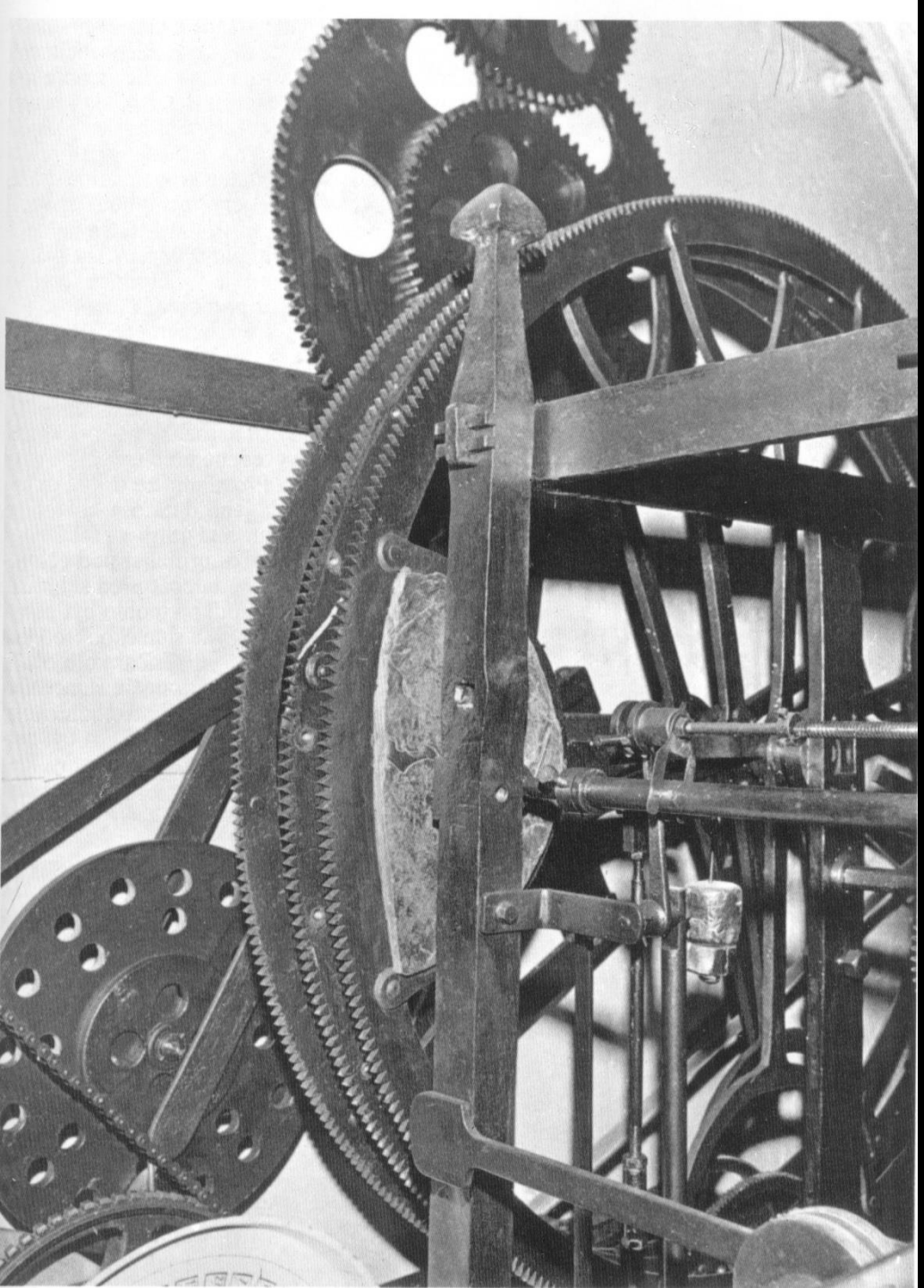

S
obratník Raka
R
M
nebeský
rovník
N
ekliptika
K
obratník Kozoroha
M′
K′
N′
R′
J
obratník Kozoroha
nebeský rovník
ekliptika
obratník Raka

View of the main clockwork from the first to the second floor.









Three large original concentric gears of the same diameter which were originally driven on one axis by three pinions, each with 24 teeth.

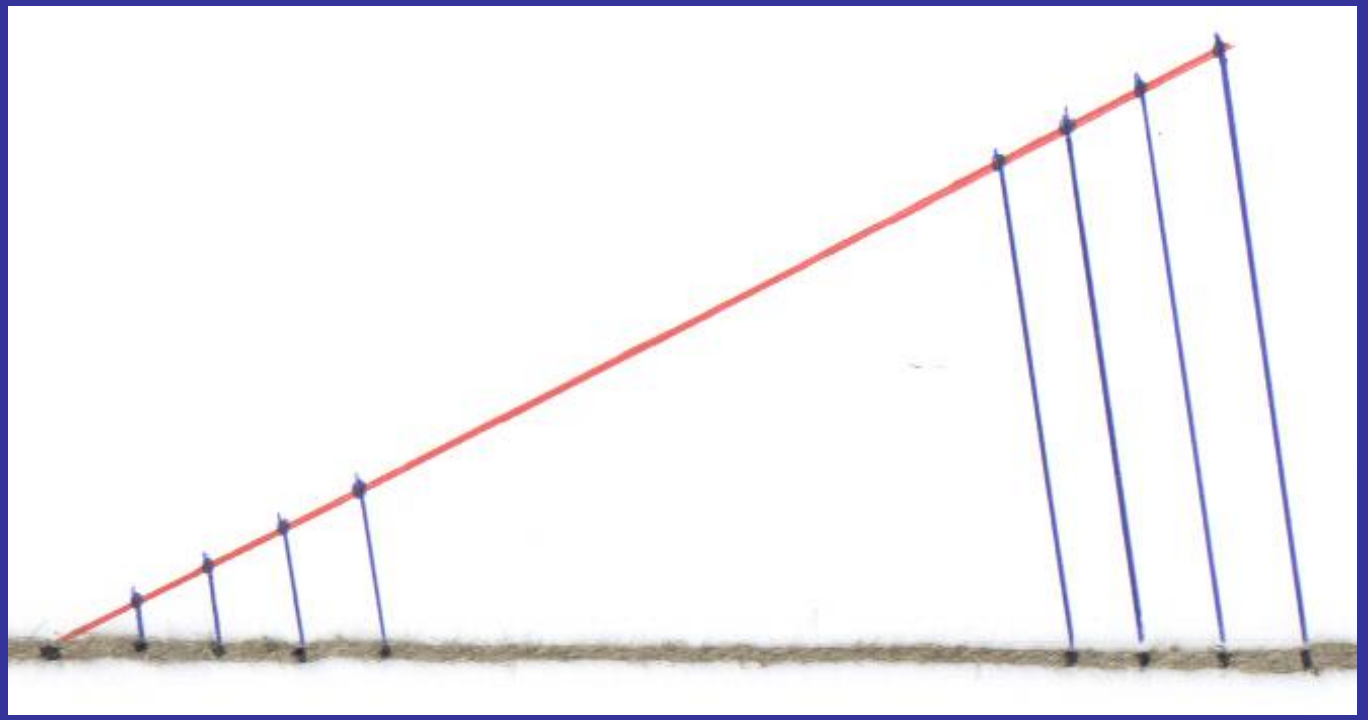
The first gear has 365 teeth and turns round the zodiac ring once per sidereal day (i.e., 23 h 56 min 4 s).

The second gear, which has 366 teeth, leads the solar pointer and turns round once per mean solar day (i.e, 24 h).

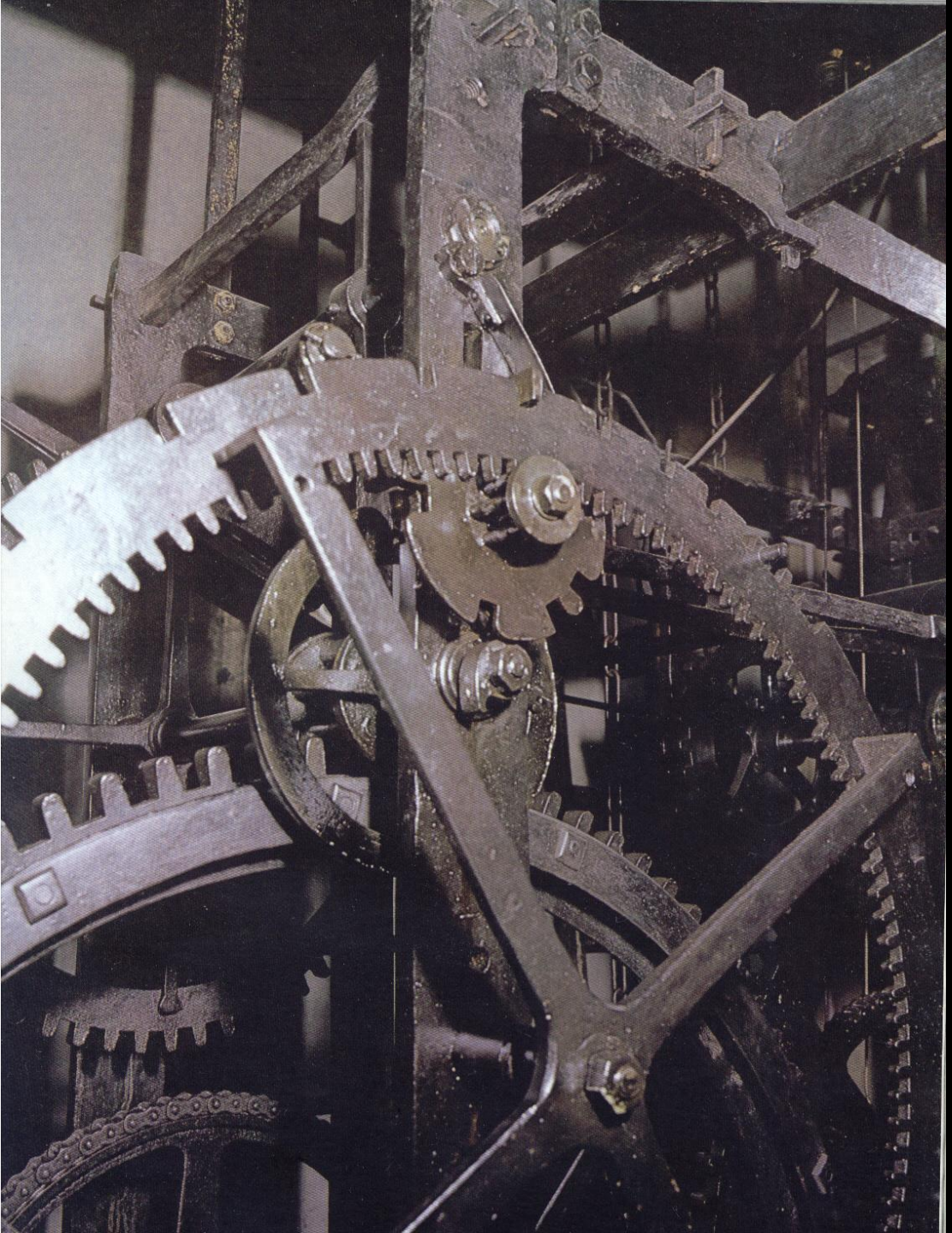
The third gear, which has 379 teeth, leads the Moon's hand and rotates according to the mean apparent motion of the Moon.

Let us ask the question, how could clockmakers in the beginning of the 15th century make such large gears with a relatively high accuracy?

Theorem 4 (Gauss). *Let p be an odd prime. Then there exists a Euclidean construction of the regular p-gon iff* $p = 2^n + 1$.



M. Křížek, F. Luca, L. Somer: *17 Lectures on Fermat numbers,* Springer, New York, 2001.



The bellworks contains a large gear with 24 slots at increasing distances. This arrangement allows for a periodic repetition of 1-24 strokes of the bell each day.

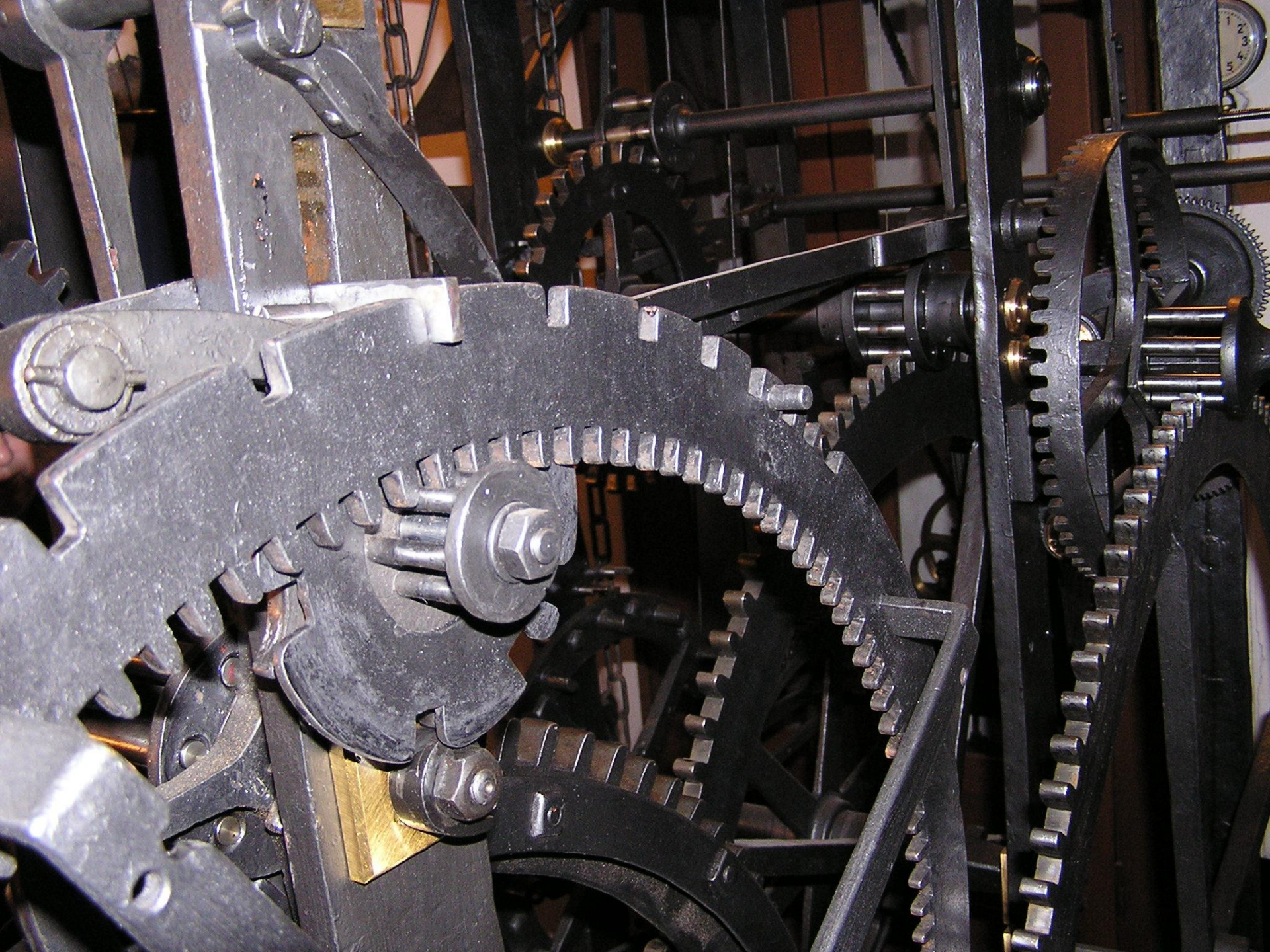
There is also a small gear whose circumference is divided by 6 slots into segments of arc lengths

$$1,\ 2,\ 3,\ 4,\ 3,\ 2\ .$$

These numbers form a period which repeats after each revolution and their sum is

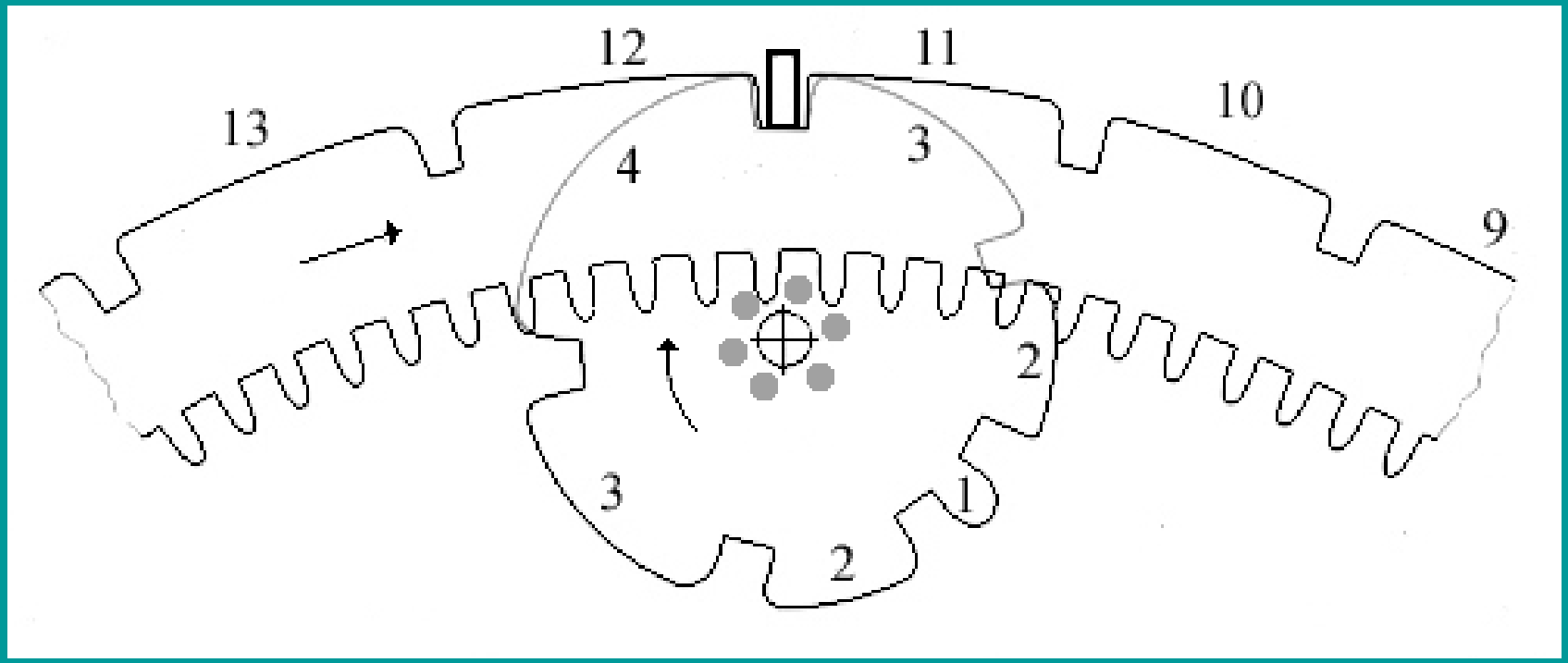
$$s = 15\ .$$

At the beginning of every hour a catch rises, both gears start to revolve and the bell chimes. The gears stop when the catch simultaneously falls down back into the slots on both gears.

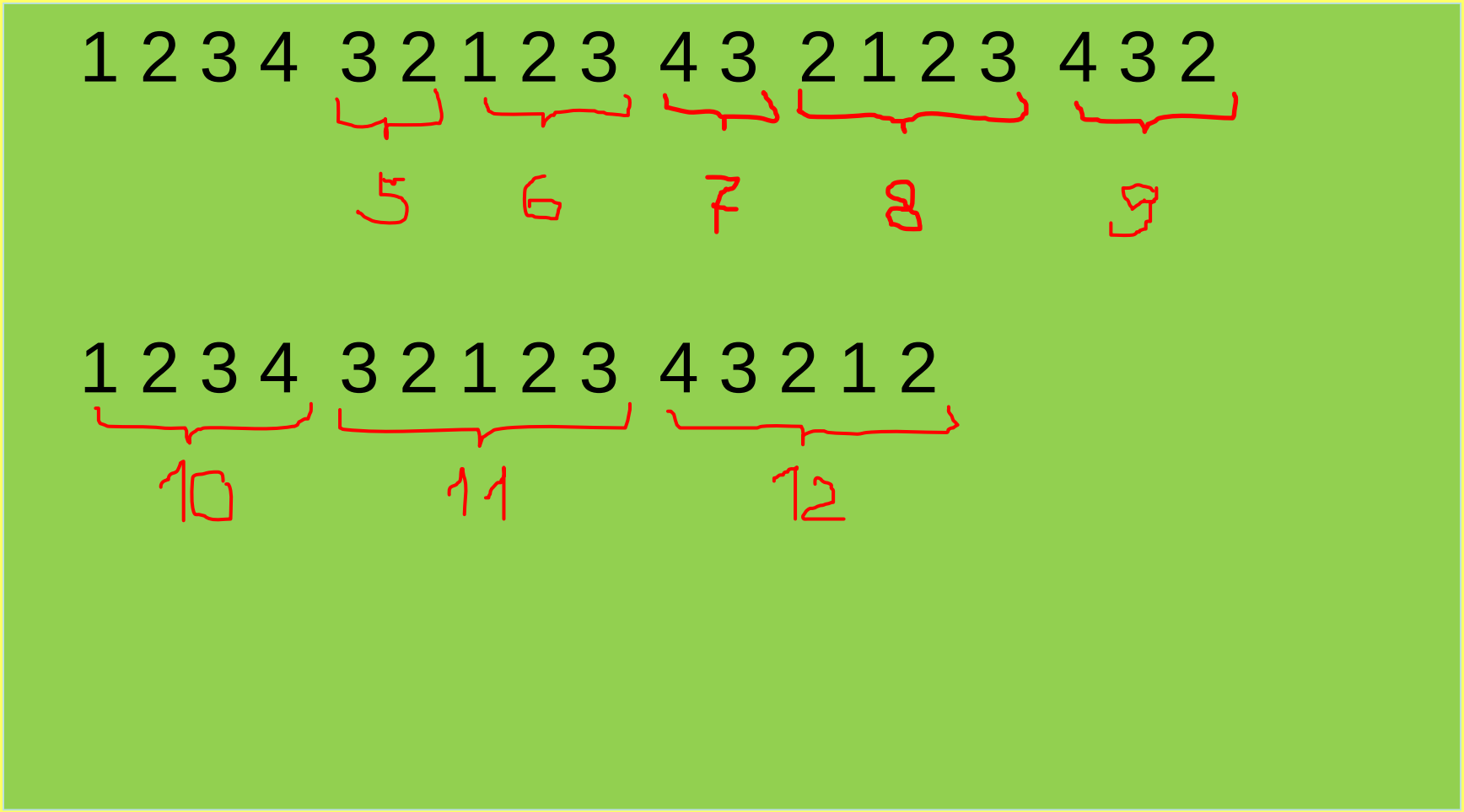


$$1+2+3+\ldots+24=300$$

the number of bell strikes every day.
Since this number is divisible by *s*=15,
the small gear is always at the same position
at the beginning of each day.



Šindel sequence 123432



Further we show how *triangular numbers*

$$T_k = 1 + 2 + \ldots + k$$

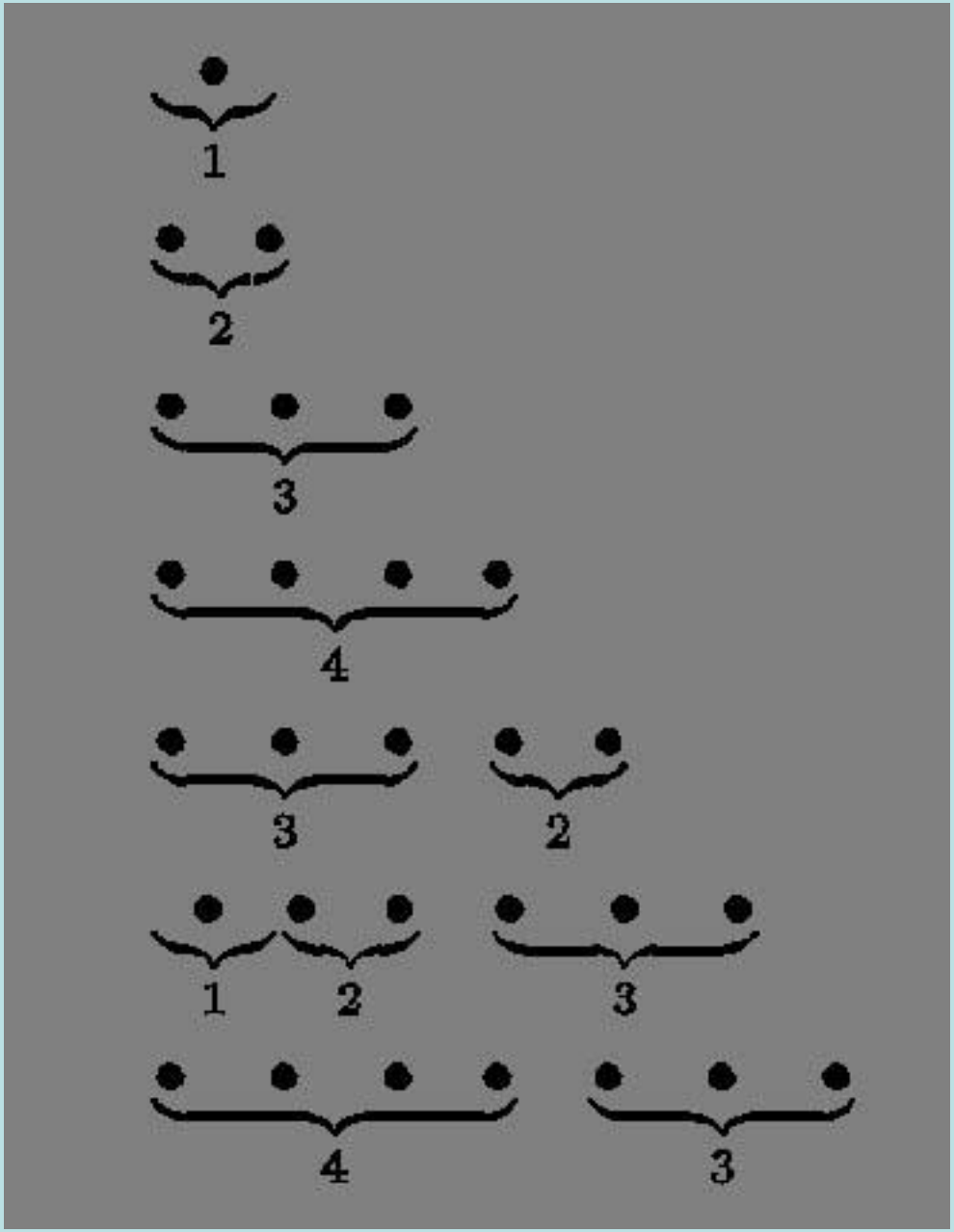
are connected with the astronomical clock.

For a periodic sequence $\{a_i\}$ we set

$$s = a_1 + a_2 + \ldots + a_p \ ,$$

where p is the period length.

T_7



A periodic sequence $\{a_i\}$ is said to be a *Šindel sequence*, if for any positive integer *k* there exists a positive integer *n* such that

$$T_k = a_1 + a_2 + \ldots + a_n$$

Theorem 5. *Let s be odd. A periodic sequence* $\{a_i\}$ *is a Šindel sequence, if for every k<(s+1)/2 there exists a positive integer n such that*

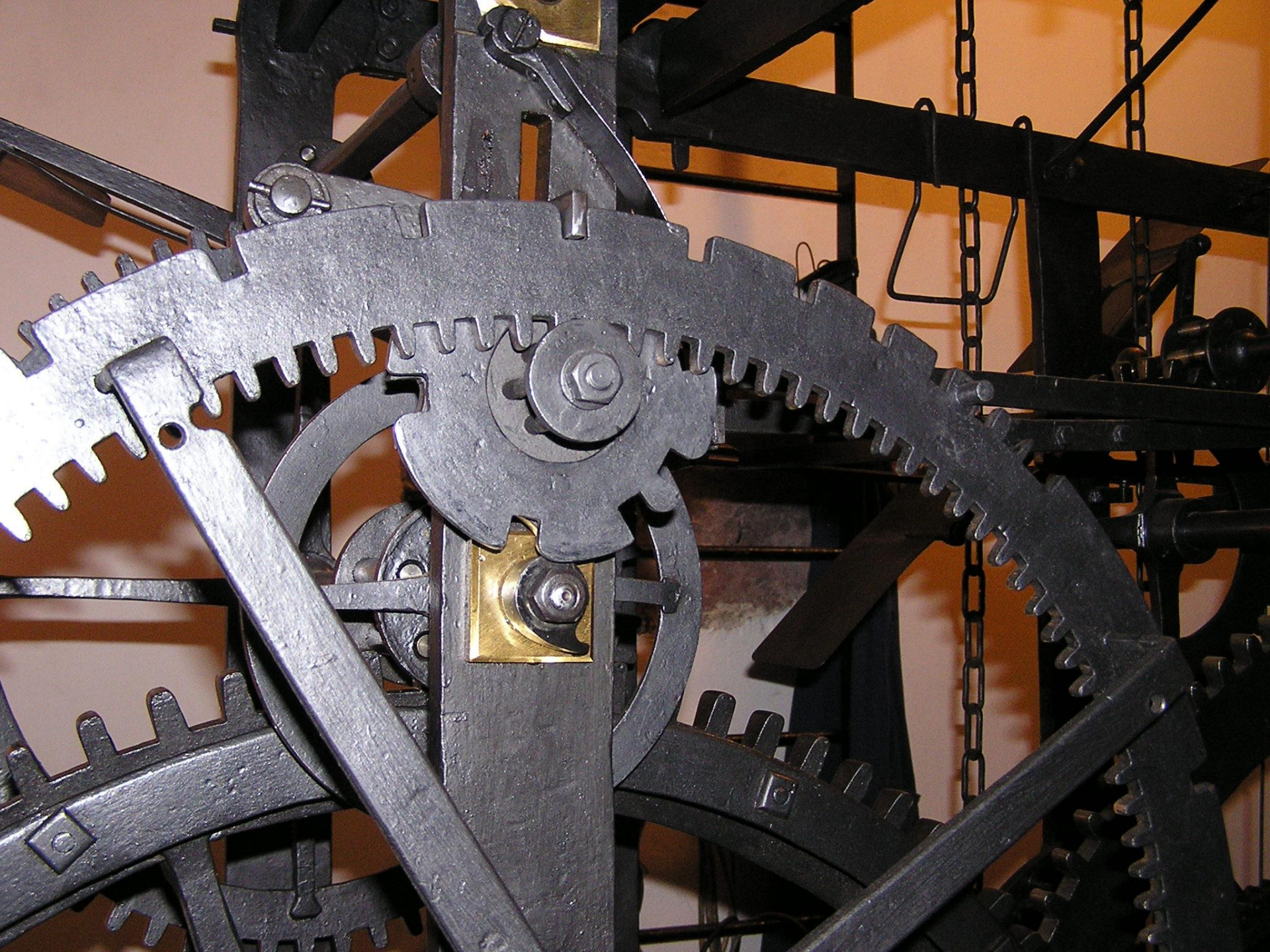
$$T_k = a_1 + a_2 + \ldots + a_n$$

Theorem 5a. *Let s be even. A periodic sequence* $\{a_i\}$ *is a Šindel sequence, if for any k<s there exists a positive integer n such that*

$$T_k = a_1 + a_2 + \ldots + a_n$$

Remark. Inequalities $k<(s+1)/2$ (resp. $k<s$) in Theorem 5 (resp. 5a) cannot be improved.

M. Křížek, L. Somer, A. Šolcová: *Jaká matematika se ukrývá v pražském orloji?,* Matematika-fyzika-informatika 16 (2006/ 2007), 129-137.



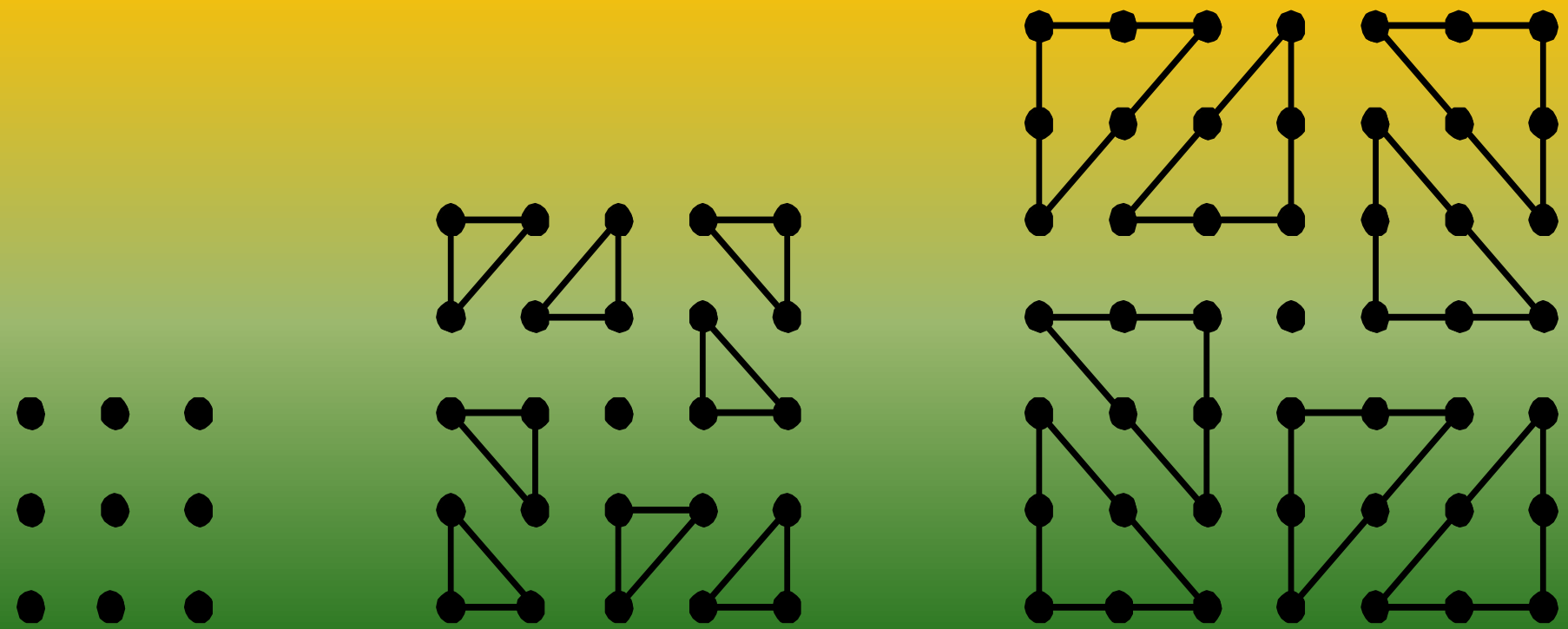
Theorem 6. *A periodic sequence* $\{a_i\}$ *is a Šindel sequence if and only if for every* $n = 1, 2, \ldots, p$ *and every* $j = 1, 2, \ldots, a_n$ *the number*

$$8(a_1 + \ldots + a_n - j) + 1$$

is a quadratic nonreziduum modulo s.

Theorem 7 (Plutarchos).
A positive integer r *is a triangular number if and only if* $8r + 1$ *is a square.*

M. Křížek, A. Šolcová, L. Somer:
What mathematics is hidden behind the astronomical clock of Prague,
Proc. of the IAU XXVI.General Assembly, Cambridge Univ. Press, 2007.



Definition.

A Šindel sequence $\{b_i\}$ with the minimal period $p + 1$ is said to be *composite,* if there exists a Šindel sequence $\{a_i\}$ and a positive integer k such that

$$a_i = b_i \quad \text{for } i = 1, \ldots, k-1$$

$$a_k = b_k + b_{k+1}$$

$$a_i = b_i \quad \text{for } i = k, \ldots, p$$

Example. The period 1, 2, 3, 2, 2, 3, 2 derived from the period 1, 2, 3, 4, 3, 2 produces a composite Šindel sequence.

Definition. A Šindel sequence is called *primitive* if it is not composite.

Theorem 8. *For any positive integer s there exists a unique primitive Šindel sequence.*

The sequence 1, 1,…, 1 is called a *trivial* Šindel sequence.

Theorem 9. *A primitive Šindel sequence is trivial if and only if s is a power of 2.*

M. Křížek, A. Šolcová, L. Somer: *Construction of Šindel sequences,* Comment. Math. Univ. Carolin. 48 *(*2007), 1-16.

Primitive Šindel sequences

s	
1	1
2	1 1
3	1 2
4	1 1 1 1
5	1 2 2
6	1 2 1 2
7	1 2 3 1
8	1 1 1 1 1 1 1 1
9	1 2 3 3
10	1 2 2 1 2 2
11	1 2 1 2 4 1
12	1 2 1 2 1 2 1 2
13	1 1 1 3 2 2 3
14	1 2 3 1 1 2 3 1
15	**1 2 3 4 3 2**
16	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
17	1 1 1 1 2 4 1 4 2
18	1 2 3 3 1 2 3 3
19	1 1 1 3 1 2 1 5 2 2
20	1 2 2 1 2 2 1 2 2 1 2 2

Theorem 10. *For any positive integer k there exists $m>k$ and a Šindel sequence $\{a_i\}$ so that*

$$a_m = k\,.$$

Example. The period

1, 2, 3, 4, 5, 3, 3, 7, 2, 3, 3, 9

for $p = 12$ and $s = 45$ yields a Šindel sequence with a relatively large element 9 .

M. Křížek, A. Šolcová, L. Somer: *Construction of the Šindel sequences,* Proc. Internat. Conf. PANM 13, Prague, 2006, 156-164.

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Šindel sequence for $s = 365$

1 2 3 4 1 2 2 3 3 4 3 5 3 5 4 1 4 5

3 5 3 4 6 2 3 5 4 1 9 1 4 1 **14** **10** 1 5

15 2 3 2 3 2 8 2 **17** 1 5 5 9 6 4 5 1 4

1 5 4 6 7 2 5 5 1 **10** 2 7 8 5 2 1 4 5

3 **12** **10** **13** 5 2 7

1793

v roce 1793. Část kolorovaného leptu Caspara Plutha podle Filipa a Františka Hegra. Pozoruhodné jsou postavičky v otevřených okénkách

1794

The height
of the townhall tower
is 59 m = 100 ells.
It was founded
in the first half
of the 14th century.

[70] Filip a František Hegerové: *Staroměstské náměstí, 1794*

May 1945